Preserving Synchronization under Characteristic Polynomial Modifications

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Abstract: In this article we present a methodology under which stability and synchronization of a dynamical master/slave system configuration are preserved under specific modifications made to its Jacobian matrix's characteristic polynomial. We propose to modify the coefficients of the associated characteristic polynomial by calculating their value to the m-th power, with m an odd, positive integer. The objective is to show that under these modifications, hyperbolic critical points are preserved along the stable and unstable manifolds. It is also shown that a consequence of this approach is the preservation of the signature of the Jacobian matrix associated with the dynamical system. To illustrate the results we present several examples of well known chaotic attractors.

Keywords: Synchronization preservation, Chaotic systems, Control, Nonlinear systems, Output feedback and Observers

1. INTRODUCTION

The study of synchronization preservation is relevant when it comes to chaos control problems. As a matter of fact, the generalized synchronization can even be derived for different systems by finding a diffeomorphic transformation such that the states of the slave system can be written as a function of the states of the master dynamics (see Femat et al. (2005) and references therein). Preservation of stability for a class of nonlinear autonomous dynamical systems has been reported in the last decades Khalil (2002). The underlying idea is to preserve the stability properties under transformation of finite-dimensional dynamical systems. In the case of linear dynamical systems there exist several results of stability preservation, for instance in Fernández-Anaya et al. (2004), stability is asymptotically preserved using transformations on rational functions in the frequency domain. The problem of stability and synchronization preservation has been recently addressed for the case of hyperbolic, nonlinear systems with chaotic dynamics in Fernández-Anaya et al. (2007) and Becker-Bessudo et al. (2008). Results reported in these articles deal with strictly linear modifications, i.e. constant term matrix multiplication. Based on these results the goal has been to develop further studies in the field of synchronization preservation for modified dynamical systems. One of the advances we have looked into has been the use of *nonlinear* modifications over the linear part of these systems. The pursuit of this line of thought has involved the development of new criteria as to the extent to which the system's stability, hyperbolic points and synchronizability are preserved under such transformations. This has in turn led to some unusual techniques in the design of state feedback controllers that would allow synchronization in such cases.

The results presented in this paper give an answer to the the problems posed above by insuring that the proposed transformation is robust, meaning that although the system has been modified it still preserves the aforementioned desired qualities (stability, synchronization, hyperbolic points). We propose to develop a methodology and criteria to extend some previous results of dynamical systems theory by preserving the signature structure of the real parts of the eigenvalues of an underlying Jacobian matrix in the equilibrium points of the dynamical system. In particular, we present a simple extension of the Local Stable-Unstable Manifold Theorem. The developed methodology is used to study the problem of preservation of synchronization in chaotic dynamical systems using strictly nonlinear modifications; specifically we propose modifications performed on the coefficients of the characteristic polynomial associated to the system's Jacobian matrix. It is based on the use of matrix theory tools, specifically, the Routh-Hurwitz theorem and the Controllable Canonical Form. Positive results seen from this methodology show that the local stability of the system around a hyperbolic equilibrium point is mainly determined by the system's linear component. We present a series of modified dynamical systems to show the preservation of the synchronization manifold.

2. MATHEMATICAL PRELIMINARIES

In this section we present the necessary definitions and results that will allow us to prove the main propositions of this paper. The results stem from the use of stability theory for dynamical systems laying in $\mathbb{R}^3$ and are focused on the task of preserving hyperbolic points under a transformation over the linear part of a vector field of a nonlinear autonomous system.

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2.1 Controllable Canonical Form

The Controllable Canonical Form (see Åström and Wittenmark (1990)) will allow us to construct a controllable matrix out of a predetermined characteristic polynomial. Consider A to be a 3×3 matrix whose characteristic polynomial is defined by the monic polynomial

$$p(s) = s^3 + a_2 s^2 + a_1 s + a_0 \tag{1}$$

where $\{a_1\} \in \mathbb{R}$ and $\{a_2, a_0\} \in \mathbb{R} - \{0\}$.

We can define its company matrix based on this as

$$A_c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} \tag{2}$$

Having done this it is possible to find a similarity matrix T defined by

$$T = [A^2C + a_2AC + a_1C \vdots AC + a_2C \vdots C] \tag{3}$$

where the column vector C is selected such that (A, C) is reachable. This implies that A can be defined as

$$A = TA_cT^{-1} \tag{4}$$

This theorem will allow us to reconstruct a new matrix after the polynomial $p(s)$ has been modified.

2.2 Associated Hurwitz Matrix

Consider $p(s)$ as in the previous subsection. The signature sequence of the spectrum of this polynomial may be determined by calculation of the main minors ($\{\Delta_1, \Delta_2, \Delta_3\}$) of its associated Hurwitz matrix

$$\hat{H} = \begin{pmatrix} a_2 & a_0 & 0 \\ 1 & a_1 & 0 \\ 0 & a_2 & a_0 \end{pmatrix} \tag{5}$$

Explicit calculation of the minors of this matrix yields

$$\begin{aligned} \Delta_1 &= a_2 \\ \Delta_2 &= a_2a_1 - a_0 \\ \Delta_3 &= a_2a_1a_0 - a_0^2 \end{aligned} \tag{6}$$

having done this we may define the *inertia*, $In(p)$, of the polynomial $p(s)$ as follows

$$In(p) = \begin{cases} \nu(p(s)) = P\left(1, \Delta_1, \dfrac{\Delta_2}{\Delta_1}, \dfrac{\Delta_3}{\Delta_2}\right) = k \\ \pi(p(s)) = 3 - P\left(1, \Delta_1, \dfrac{\Delta_2}{\Delta_1}, \dfrac{\Delta_3}{\Delta_2}\right) = 3 - k \\ \delta(p(s)) = 0 \end{cases} \tag{7}$$

The function $P(\{\Delta\})$ denotes the number of sign consistencies in the appropriate sequence (see Lancaster and Tismenetsky (1985)). The element $\pi(p(s))$ represents the number of eigenvalues with real positive coefficients, $\nu(p(s))$ the number of eigenvalues with negative real part and $\delta(p(s))$ the number of coefficients with real part equal to zero. This theorem, under the proposed restrictions in the following section, will allow us to prove the results presented in this paper regarding the extension of the Local Stable-Unstable Manifold Theorem.

3. STABILITY PRESERVATION

For the following discussion consider the dynamical system described by

$$\dot{x} = f(x)$$

where $x \in \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a continuous differentiable function of its argument. Let $A = \left.\frac{\partial f}{\partial x}\right|_{x_0}$ be the Jacobian matrix associated with f evaluated at an equilibrium point x_0.

Lemma 1. If $p(s)$, the characteristic polynomial of our Jacobian matrix, is a real third degree polynomial, and there are no zeros in the sequence

$$\Delta_1, \Delta_2, \Delta_3 \tag{8}$$

of leading principal minors of the corresponding Hurwitz matrix, then $p(s)$ has no pure imaginary zeroes, i.e. the equilibrium point is a hyperbolic point with inertia

$$\begin{aligned} \nu(p(s)) &= k \\ \pi(p(s)) &= 3 - k \\ \delta(p(s)) &= 0 \end{aligned} \tag{9}$$

Now consider the modified $p(s)$ polynomial as

$$p_m(s) = s^3 + a_2^m s^2 + a_1^m s + a_0^m \tag{10}$$

with m an odd, positive integer i.e. $m = 2r + 1$. For $r \in \mathbb{Z}^+$, then the inertia of this new polynomial's associated Hurwitz matrix is the same as the original polynomial's

$$\begin{aligned} \nu(p_m(s)) &= \nu(p(s)) = k \\ \pi(p_m(s)) &= \pi(p(s)) = 3 - k \\ \delta(p_m(s)) &= \delta(p(s)) = 0 \end{aligned} \tag{11}$$

Proof

Looking at the main minors of the original Hurwitz matrix as we defined them in section 2 we have the negative inertia of the matrix determined by

$$\nu(p(s)) = P\left(1, \Delta_1, \frac{\Delta_2}{\Delta_1}, \frac{\Delta_3}{\Delta_2}\right) \tag{12}$$

and the negative inertia of the modified matrix by

$$\nu(p_m(s)) = P\left(1, a_2^m, \frac{a_2^m a_1^m - a_0^m}{a_2^m}, \frac{a_2^m a_1^m a_0^{2m} - a_0^{2m}}{a_2^m a_1^m - a_0^m}\right)$$
$$= P\left(1, \Delta_{1m}, \frac{\Delta_{2m}}{\Delta_{1m}}, \frac{\Delta_{3m}}{\Delta_{2m}}\right) \quad (13)$$

Signature preservation for Δ_1 is trivial under the proposed transformation. As for the values of Δ_2 and Δ_3, they depend solely on the subtraction of two terms and noting that $(a_1^m a_2^m \cdots a_n^m) = (a_1 a_2 \cdots a_n)^m$, choosing m as an odd, positive integer will insure that for arbitrary unsigned values of two sequences of real coefficients $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_l$ that follow

$$(a_1 a_2 \cdots a_n)^m > (b_1 b_2 \cdots b_l)^m \iff \left(\frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_l}\right) > 1 \quad (14)$$

calculating the logarithm on both sides of the second inequality we find

$$\log\left(\frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_l}\right) > 0 \quad (15)$$

Thus showing that the preservation of our original inequality is independent from the magnitude of m. This leads us to conclude that the sign structure of function $P(\Delta)$ will remain unchanged after the modification, i.e. $\mathrm{sgn}(\Delta_i) = \mathrm{sgn}(\Delta_{im})$ *for* $i = 1, 2, 3$.

■

Using these results we can reconstruct our modified matrix A_m as

$$A_m = T A_{cm} T^{-1} = T \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0^m & -a_1^m & -a_2^m \end{pmatrix} T^{-1} \quad (16)$$

4. LOCAL STABLE-UNSTABLE MANIFOLD THEOREM EXTENSION

In this section, we show a simple extension of the Local Stable-Unstable Manifold Theorem, using the tools presented in section 2. The results will be used in section 6, where we will present some examples on preservation of synchronization in dynamical systems.

The following proposition is a simple extension of the Local Stable-Unstable Manifold Theorem under perturbations suffered by the characteristic polynomial associated to the dynamical system's Jacobian matrix A of the vector field $f(x)$.

It should be noted that this proposition is an alternative result to those presented in Fernandez-Anaya et al. (2008).

Proposition 2. Let E be an open subset of $\mathbb{R}^3$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system $\dot{x} = f(x) = Ax + g(x)$. Suppose that $f(0) = 0$ and that $A = Df(0)$ has k eigenvalues with negative real part and $3-k$ eigenvalues with positive real part, i.e., the origin is an hyperbolic fixed point. There exists a k-dimensional differentiable manifold S tangent to the stable subspace E^S and a $3-k$ dimensional differentiable manifold W tangent to the unstable subspace E^W of the linear system $\dot{x} = Ax$ at 0. Then for each value m, as previously defined, there exists a k-dimensional differentiable manifold S_m tangent to the stable subspace E_m^S of the linear system $\dot{x} = A_m x$ at 0 such that for all $t \geq 0$, $\phi_{P,t}(S_m) \subset S_m$ and for all $x_0 \in S_m$,

$$\lim_{t\to\infty} \phi_{m,t}(x_0) = 0,$$

where $\phi_{m,t}$ be the flow of the nonlinear system $\dot{x} = A_m x + g(x)$; and there exists a $3-k$ dimensional differentiable manifold W_m tangent to the unstable subspace E_m^W of $\dot{x} = A_m x$ at 0 such that for all $t \leq 0$, $\phi_{m,t}(W_m) \subset W_m$ and for all $x_0 \in W_m$,

$$\lim_{t\to-\infty} \phi_{m,t}(x_0) = 0.$$

An interesting property is that Proposition 2 is valid for each $\overline{g} \in C^1(E)$ such that $\dot{x} = \overline{f}(x) = A_m x + \overline{g}(x)$ and

$$\frac{\|\overline{g}(x)\|_2}{\|x\|_2} \to 0 \text{ as } \|x\|_2 \to 0.$$

In consequence this transformation preserves hyperbolic points in nonlinear systems and dimension of the stable and unstable manifolds, i.e, an hyperbolic nonlinear system ($\dot{x} = Ax + \overline{g}(x)$) is mapped in a hyperbolic nonlinear systems ($\dot{x} = A_m x + \overline{g}(x)$), and $\dim S = \dim S_m$ and $\dim W = \dim W_m$.

Sketch of Proof:

Consider a 3×3 Jacobian matrix A associated to our dynamical system with a characteristic polynomial $p(s) = s^3 + a_2 s^2 + a_1 s + a_0$, which has k roots with negative real part and $3-k$ roots with positive real part. Then we define our modified polynomial as $p_m(s) = s^3 + a_2^m s^2 + a_1^m s + a_0^m$.

Since m is an odd integer we can assure that there are no sign changes in the coefficients of our new polynomial and by using Lemma 1, we can assure the matrix A_m has k eigenvalues with negative real part and $3-k$ eigenvalues with positive real part. Since the dimensions of each manifold associated with the eigenvalues of our new system have not changed, the result is a consequence of the Local Stable-Unstable Manifold Theorem. ■

Remark 3. It is however important to consider that the dynamics of the system are deeply related to the magnitude ratios between eigenvalues and their original order in A and as such significant alterations in their magnitude should be considered carefully when selecting the value of m. As a way of regulating these values we make use of the results presented in Becker-Bessudo et al. (2008) to modulate the resultant system thus allowing us carry out simulations.

The relevance of this proposition resides on the fact that critical hyperbolic points are preserved using nonlinear transformations on the characteristic polynomial $p(s)$. As a consequence of this we established criteria which allow us to preserve the signature of the associated Jacobian matrix.

5. PRESERVATION OF SYNCHRONIZATION IN MODIFIED SYSTEMS

In this section we show how it is possible to preserve synchronization after the system's eigenvalues have been modified under the action of the class of transformation on the linear part of the nonlinear system described in the last section.

Consider the following *3*-dimensional systems in a master-slave configuration, where the master system is given by

$$\dot{x} = Ax + g(x)$$

and the slave system is

$$\dot{y} = Ay + f(y) + u(t)$$

where $A \in R^{3\times3}$ is a constant matrix, $f, g : R^3 \to R^3$ are continuous nonlinear functions and $u \in R^3$ is the control input. The problem of synchronization considered in this section is the complete-state exact synchronization. That is, the master system and the slave system are synchronized by designing an appropriate nonlinear state feedback control $u(t)$ which is attached to the slave system such that

$$\lim_{t\to\infty} \|y(t) - x(t)\| \to 0$$

where $\|\cdot\|$ is the Euclidean norm of a vector.

Considering the error state vector $e = y - x \in R^3$, $f(y) - g(x) = L(x, y)$ and an error dynamics equation

$$\dot{e} = Ae + L(x, y) + u(t).$$

Based in the active control approach Bai and Lonngren (2000), to eliminate the nonlinear part of the error dynamics, and choosing $u(t) = Bv(t) - L(x, y)$, where B is a constant gain matrix which is selected such that (A, B) be controllable, we obtain

$$\dot{e} = Ae + Bv(t).$$

Notice that the original synchronization problem is equivalent to the problem of stabilizing the zero-input solution of the last system by a suitable choice of the state feedback control.

Since the pair (A, B) is controllable one such suitable choice for state feedback is a linear-quadratic state-feedback regulator Anderson and Moore (1990), which minimizes the quadratic cost function

$$J(u(t)) = \int_0^\infty \left(e(t)^\top Qe(t) + v(t)^\top Rv(t)\right) dt$$

where Q and R are positive semi-definite and a positive definite weighting matrices, respectively. The state-feedback law is given by $v = -Ke$ with $K = R^{-1}B^\top S$ and S the solution to the Riccati equation

$$A^\top S + SA - SBR^{-1}B^\top + Q = 0.$$

This state-feedback law renders the error equation to $\dot{e} = (A - BK)e$, with $(A - BK)$ a Hurwitz matrix [1]. The linear quadratic regulator (LQR) is a well-known design technique that provides practical feedback gains Anderson and Moore (1990). An interesting property of (LQR) is robustness.

Now consider m an odd integer, and suppose that the following two 3-dimensional systems are chaotic for some $f, g : R^3 \to R^3$ continuous nonlinear functions and $\widehat{u} \in R^3$ is the control input.

$$\begin{aligned}\dot{x} &= A_m x + g(x)\\ \dot{y} &= A_m y + f(y) + \widehat{u}(t)\end{aligned}$$

where A_m was constructed using the Controllable Canonical Form as presented in section 2.

Now, suppose that $\widehat{u}(t) = \widehat{\theta}(t) - L(x, y)$ stabilizes the zero solution of the error dynamics system, where $\widehat{\theta}(t) = -(BK_m)e$, i.e., the resultant system

$$\begin{aligned}\dot{e} &= A_m e + \widehat{\theta}(t)\\ \dot{e} &= (A_m - BK_m)\, e\end{aligned}$$

is asymptotically stable. The process to find matrix K_m used for the control input was defined using the matrix K derived from solving the Riccati equation and using the Controllable Canonical Form as follows. Having found K such that $A - BK$, taking $B = I$, is a Hurwitz matrix. We can find the company and similarity matrices as we did with matrix A

$$A - K = T_0(A - K)_c T_0^{-1} \tag{17}$$

next we find the modified matrix

$$(A - K)_m = T_0(A - K)_{cm} T_0^{-1} \tag{18}$$

and finally we define K_m as

$$\begin{aligned}-K_m &\equiv (A - K)_m - A_m\\ A_m - K_m &= (A - K)_m\end{aligned} \tag{19}$$

with both $A_m - K_m$ and $(A - K)_m$ Hurwitz matrices.

6. SIMULATIONS

6.1 The Rössler Attractor

The original dynamical system of what is know as the Rössler Attractor is defined by

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3\\ \dot{x}_2 &= x_1 + \frac{1}{10}x_2\\ \dot{x}_3 &= \frac{1}{10} + x_3(x_1 - 14)\end{aligned}$$

[1] A Hurwitz matrix is a matrix for which all its eigenvalues are such that their real part is strictly less than zero

which has a chaotic attractor. In order to observe synchronization behavior we present two Rössler attractors arranged as a master/slave configuration. The initial conditions for the two systems are different.

The master system is given by the aforementioned equations and the slave system is a copy of the master system plus a control function $u(t)$ to be determined in order to synchronize the two systems.

$$\begin{aligned}\dot{y}_1 &= -y_2 - y_3 + u_1(t)\\ \dot{y}_2 &= y_1 + \frac{1}{10}y_2 + u_2(t)\\ \dot{y}_3 &= \frac{1}{10} + y_3(y_1 - 14) + u_3(t)\end{aligned}$$

Considering the errors $e_1 = y_1 - x_1$, $e_2 = y_2 - x_2$, $e_3 = y_3 - x_3$, then the error dynamics equations may be written as

$$\begin{aligned}\dot{e}_1 &= -e_2 - e_3 + u_1(t)\\ \dot{e}_2 &= e_1 + \frac{1}{10}e_2 + u_2(t)\\ \dot{e}_3 &= \frac{1}{10} + y_3 y_1 - x_3 x_1 - 14(e_3) + u_3(t)\end{aligned}$$

Introducing the Jacobian (A) and non-linear terms (L) matrices

$$A = \begin{pmatrix} 0 & -1 & -1\\ 1 & \frac{1}{10} & 0\\ 0 & 0 & -14\end{pmatrix},\ L(x,y) = \begin{pmatrix} 0\\ 0\\ \frac{1}{10} + y_3 y_1 - x_3 x_1\end{pmatrix},$$

$$u = \begin{pmatrix} u_1(t)\\ u_2(t)\\ u_3(t)\end{pmatrix}$$

The spectrum of A should be noted as it proves the system is hyperbolic and will allow us to compare the sign structure after the system has been modified

$$\sigma(A) = \begin{pmatrix} 0.05 + .9987i\\ 0.05 - 0.9987i\\ -14\end{pmatrix}$$

We select the matrix B such that (A, B) is controllable: $B = I$. Now the LQR controller is obtained by using weighting matrices $Q = I$ and $R = B^\top B = I$. The state feedback matrix is given by

$$K = \begin{pmatrix} 1.0239 & 0.0269 & -0.0678\\ 0.0269 & 1.0775 & 0.0028\\ -0.0678 & 0.0028 & 0.0403\end{pmatrix}$$

In Figure 1 the trajectories for the solution of the master system and slave system are shown. In Figure 2 the absolute value for the errors between the master and slave systems are shown in a semi-logarithmic plot to emphasize the fact that the error converges to zero and therefore the synchronization between the master and slave systems is successfully achieved.

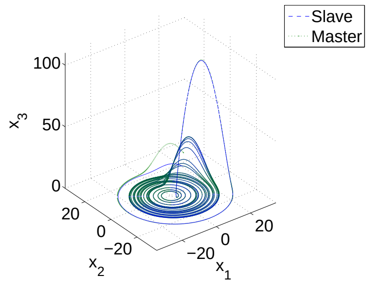

Fig. 1. Original Rössler attractor showing synchronization between master and slave systems (initial conditions $x_1 = 20, x_2 = 16, x_3 = 12$ and $y_1 = 17, y_2 = 13, y_3 = 5$ respectively).

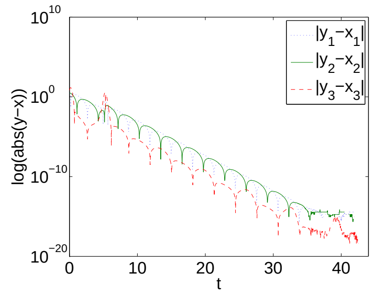

Fig. 2. Magnitude of error $|e| = |y - x|$ between original Rössler master and slave systems.

6.2 The modified Rössler Attractor

The following examples show the modifications performed on the Rössler attractor using $m = 3$. In order to allow for simpler simulations we have modulated the matrix resulting from the polynomial modification using the results presented in Becker-Bessudo et al. (2008).

$$\dot{x} = (MA_3)x + \left[0\ 0\ \frac{1}{10} + x_3 x_1\right]^\top,$$

$$\dot{y} = (MA_3)y + \left[0\ 0\ \frac{1}{10} + y_3 y_1\right]^\top + u(t)$$

Considering the error vector $e = y - x$, then the error dynamics may be written as

$$\dot{e} = M(A_3)e + L(x,y) + u(t)$$

with $u = -L(x,y) + v$ and $v = -(MBK_3)e$

Using the characteristic polynomial of the Jacobian matrix we can define the company matrix A_c, with $C = [0\ 0\ 1]^\top$, and the similarity matrix T

$$A_c = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -14 & 0.4 & -13.9\end{pmatrix},\ T = \begin{pmatrix} 0.1 & -1 & 1\\ -1 & 0 & -0.1\\ 0 & 0 & 1\end{pmatrix},$$

after modifying the coefficients of the company matrix by using $m = 3$ we have the new Jacobian matrix

$$TA_{c3}T^{-1} = A_3 = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0.1 & 0 \\ 267.508 & 85.0318 & -2685.72 \end{pmatrix},$$

notice the signature structure of the spectrum A_3 is identical to the original system's

$$\sigma(A_3) = \begin{pmatrix} 0.0002 + 1.0108i \\ 0.0002 - 1.0108i \\ -2685.6 \end{pmatrix}$$

next we select M (as established by Proposition 3.1 in Becker-Bessudo et al. (2008)) so that the resultant eigenvalues are similar to those of the original system

$$M = \begin{pmatrix} 0.9079 & 0.0817 & -0.0003 \\ -0.0791 & 0.8921 & 0 \\ 0.08740 & 0.0363 & 0.005 \end{pmatrix}$$

finally arriving at the new Jacobian matrix

$$MA_3 = \begin{pmatrix} -0.0083 & -0.9284 & -0.005 \\ 0.9 & 0.1709 & -0.0003 \\ 1.3651 & 0.3386 & -13.4286 \end{pmatrix},$$

following the established procedure in section 5 we can find K_3 and determine $u = -(MBK_3)e - L(x, y)$.

$$MBK_3 = \begin{pmatrix} 43.7582 & -12.9144 & -1.0121 \\ -3.7635 & 2.0868 & 0.0902 \\ -632.233 & 192.4037 & 14.0325 \end{pmatrix},$$

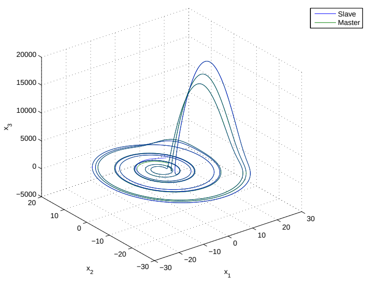

Fig. 3. Master and slave system (initial conditions $x_1 = 5, x_2 = 2, x_3 = 9$ and $y_1 = 7, y_2 = 5, y_3 = 3$, respectively) synchronization of modified Rössler attractor (with $m = 3$).

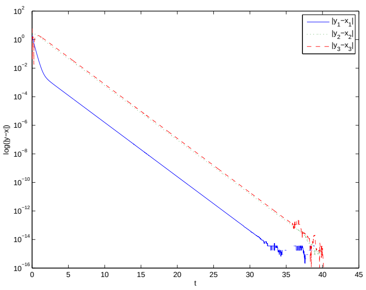

Fig. 4. Magnitude of error $|e| = |y - x|$ between master and slave system ($m = 3$).

In Figure 4 we have the absolute error of the master/slave system configuration, this time for the modified system, and again we see that there is an effective convergence to zero, thus proving that synchronization is preserved under the transformation.

7. CONCLUSIONS

The preservation of hyperbolic behavior in chaotic synchronization is studied from an extension of the Local Stable-Unstable Manifold Theorem based in the preservation of the signature of the linear part of the vector fields in nonlinear dynamical systems. Given this basic premise we set specific constraints for the possible nonlinear modifications performed on the system. Having established a viable transformation, i.e. power modification of the coefficients of the characteristic polynomial associated with the Jacobian matrix, we designed a control law that would allow us to preserve synchronization under the same transformation. The effectiveness of the proposed method can be ascertained by the results seen in the simulations. Future work focused on the extension of this line of thought, nonlinear modification of a system's characteristic polynomial, will be based on modification under positive, real numbers as well as the use of fewer inputs for the controller.

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