

Relativistic bulk viscosity in the relaxation time approximation: a chaotic velocities approach

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Abstract. In this short note, the bulk viscosity for a high temperature dilute gas is calculated by applying the Chapman-Enskog method within Marle's relaxation time approximation. The expression for the stress-tensor established in Ref.[1], using explicitly the concept of chaotic velocity, is used in order to obtain the transport coefficient. The result is compared with previous expressions obtained by other authors using similar methods and emphasis is made on the agreement when a corrected relaxation parameter is considered.

1. Introduction

Transport coefficients are of current interest in high temperature gases, when the thermal energy becomes comparable or exceeds the rest energy of the individual particles. Other authors have calculated these gas parameters using various methods within kinetic theory. In this work we introduce the simplest model for the collisional term of the relativistic Boltzmann's equation, namely Marle's model, in the particle's frame and follow a direct method analogous to the one employed in the BGK approximation. Following such procedure, one can obtain directly the part of the distribution function which is coupled to the derivatives of the hydrodynamic velocity, making possible the calculation of the corresponding bulk viscosity. The expression for the stress tensor here considered corresponds to the one established in Ref.[1] by means of explicitly introducing the concept of chaotic velocity in relativistic kinetic theory. In such formalism, Lorentz transformations are used in order to write thermodynamic fluxes in an arbitrary reference frame in terms of integrals which can be calculated in a comoving one.



An analytic expression for the coefficient, which vanishes in the non-relativistic low temperature limit, is established following the steps described above. The result is compared with values obtained by Cercignani & Kremer [2] using the complete collision kernel and a procedure akin to a Maxwellian iteration, within Marle's approximation in the method of moments. The results obtained in the present work are consistent with those presented in Ref.[2] and a better adjustment in the ultra-relativistic case is archived when the adjusted relaxation time established in Ref.[3] is considered. The coefficient here obtained, together with previous ones for other fluxes may be useful for addressing hot dilute gases in dissipative regimes.

In order to accomplish the task described above we have divided the rest of the work in three sections. In Section 2 we state the relativistic Boltzmann equation and outline the steps leading to the deviation from the equilibrium distribution function within the Chapman-Enskog hypothesis. Section 3 is devoted to the explicit calculation of the bulk viscosity coefficient. A brief discussion and final remarks are included in Section 4.

2. The relativistic Boltzmann equation within Marle's approximation

The starting point of the calculation is the relativistic Boltzmann equation with a relaxation time approximation:

$$v^\alpha f_{,\alpha} = -\frac{f - f^{(0)}}{\tau} \quad (1)$$

where v^α is the molecular velocity measured in an arbitrary reference frame and τ is a parameter related to characteristic time (see Ref. [3]) in which collisions relax the distribution function f to its equilibrium value $f^{(0)}$. For the relativistic dilute gas, $f^{(0)}$ is given by

$$f^{(0)} = \frac{n}{4\pi c^3 z K_2\left(\frac{1}{z}\right)} e^{\frac{\mathcal{U}_\beta v^\beta}{zc^2}} \quad (2)$$

where n is the particle number density and \mathcal{U}_β the hydrodynamic velocity. An account of the basic properties of the Jüttner distribution function can be found in Refs.[2, 4, 5, 6]. These, together with the temperature T , which corresponds to the local temperature, measured in the comoving frame of any volume element of the fluid and is included in the relativistic parameter $z = k_B T/mc^2$, are the state variables describing the gas. Also, c is the speed of light, k_B is Boltzmann's constant and K_n the n -th modified Bessel function of the second kind.

Introducing the Chapman-Enskog hypothesis, $f = f^{(0)} + f^{(1)}$ in Eq. (1) and following the standard procedure (see for example Ref.[7]) one obtains, for the non-

equilibrium part of the distribution function

$$f^{(1)} = -\tau f^{(0)} \left\{ v^\beta h_{\beta}^{\alpha} \left[\frac{n_{,\alpha}}{n} + \frac{T_{,\alpha}}{T} \left(1 - \frac{v^\mu \mathcal{U}_\mu}{zc^2} - \frac{\mathcal{G}\left(\frac{1}{z}\right)}{z} \right) + \frac{v_\mu}{zc^2} \mathcal{U}_{;\alpha}^\mu \right] \right. \\ \left. + \left(\frac{v^\beta \mathcal{U}_\beta}{c^2} \right) \left[\mathcal{U}_{;\mu}^\mu + \frac{k_B}{C_n} \mathcal{U}_{;\nu}^\nu \left(1 - \frac{v^\mu \mathcal{U}_\mu}{zc^2} - \frac{\mathcal{G}\left(\frac{1}{z}\right)}{z} \right) + \frac{p_{,\nu} h^{\mu\nu} v_\mu}{mn \mathcal{G}\left(\frac{1}{z}\right) zc^2} \right] \right\} \quad (3)$$

where $\mathcal{G}\left(\frac{1}{z}\right) = K_3\left(\frac{1}{z}\right)/K_2\left(\frac{1}{z}\right)$, $p = nkT$ is the hydrostatic pressure and C_n is the heat capacity. In order to obtain Eq. (3) use has been made of the following decomposition for the molecular velocity

$$v^\alpha = v^\beta h_{\beta}^{\alpha} - \left(\frac{v^\beta \mathcal{U}_\beta}{c^2} \right) \mathcal{U}^\alpha \quad (4)$$

where $h^{\mu\nu} = \eta^{\mu\nu} + \mathcal{U}^\mu \mathcal{U}^\nu / c^2$ is the so-called spatial projector. Also, the Euler equations have been used in order to write proper time derivatives in terms of spatial gradients.

Equation (4) constitutes an explicit expression for the first order in the gradients correction to the equilibrium distribution function. This attractive feature, the fact that the distribution function can be solved for analytically, makes this procedure particularly simple and useful in order to establish the structure of constitutive equations in a somewhat straightforward manner and the basic dependence of transport coefficients with state variables.

In order to address viscous dissipation, we start by noticing that the vector terms, containing $n_{,\alpha}$, $T_{,\alpha}$ and $p_{,\alpha}$ will not contribute to the stress tensor. This can be easily seen by examining the parity in the integral leading to the momentum flux average. Also, Curie's principle states that only forces and fluxes of the same tensor rank are coupled in constitutive equations. Thus, the part of the distribution function that will contribute to viscous dissipation is given by

$$-\tau f^{(0)} \left\{ h_{\beta}^{\alpha} \frac{v^\beta v_\mu}{zc^2} \mathcal{U}_{;\alpha}^\mu + \left(\frac{v^\mu \mathcal{U}_\mu}{c^2} \right) \left[1 + \frac{k_B}{C_n} \left(1 - \frac{v^\beta \mathcal{U}_\beta}{zc^2} - \frac{\mathcal{G}\left(\frac{1}{z}\right)}{z} \right) \right] \mathcal{U}_{;\nu}^\nu \right\}. \quad (5)$$

In particular, writing $\mathcal{U}_{;\alpha}^\mu$ as the sum of a symmetric traceless part, an antisymmetric part and a scalar part, the contribution arising from the first term in Eq. (5) to the trace of the stress tensor is given by

$$\frac{1}{3} \frac{1}{zc^2} h_{\beta}^{\alpha} \eta_{\alpha}^{\mu} v_\mu v^\beta \mathcal{U}_{;\ell}^\ell \quad (6)$$

where $\ell = 1, 2, 3$, and thus, the part of the distribution function proportional to the divergence of the hydrodynamic velocity is given by

$$f_b^{(1)} = -\tau f^{(0)} \left\{ \frac{1}{3zc^2} h_{\beta}^{\alpha} \eta_{\alpha}^{\mu} v_\mu v^\beta + \left(\frac{v^\beta \mathcal{U}_\beta}{c^2} \right) \left[1 + \frac{k_B}{C_n} \left(1 - \frac{v^\beta \mathcal{U}_\beta}{zc^2} - \frac{\mathcal{G}\left(\frac{1}{z}\right)}{z} \right) \right] \right\} \mathcal{U}_{;\ell}^\ell \quad (7)$$

where $\ell = 1, 2, 3$. As extensively discussed in Refs. [1, 8], the invariants included in the expression above can be calculated in a local rest frame moving with the fluid element

where the molecules' velocities are given by their chaotic, or peculiar, component which we denote by $K^\mu = \gamma_k [\vec{k}, c]$ such that

$$\frac{1}{3} \frac{1}{z c^2} h_\beta^\alpha h_\alpha^\mu v_\mu v^\beta = \frac{k^2 \gamma_k^2}{3 z c^2} \quad (8)$$

and

$$v^\mu \mathcal{U}_\mu = -\gamma_k c^2. \quad (9)$$

Chaotic velocity refers to molecular velocity measured by an observer comoving with the local hydrodynamic velocity. This variable allows for the calculation of invariant integrals in phase space and for a clear definition of state variables and dissipative fluxes in a representation analogous to the usual one in the non-relativistic case [9].

Introducing these expressions in Eq. (7) one obtains

$$f_b^{(1)} = -\tau f^{(0)} \left\{ \frac{k^2 \gamma_k^2}{3 z c^2} - \gamma_k \left[1 + \frac{k_B}{C_n} \left(1 - \frac{v^\beta \mathcal{U}_\beta}{z c^2} - \frac{\mathcal{G} \left(\frac{1}{z} \right)}{z} \right) \right] \right\} \mathcal{U}_{;l}^\ell \quad (10)$$

which will be used in the following section in order to obtain the bulk viscosity within this framework. Notice that, since

$$\frac{k_B}{C_n} = \frac{z^2}{\left[1 + 5z \mathcal{G} \left(\frac{1}{z} \right) - \mathcal{G}^2 \left(\frac{1}{z} \right) - z^2 \right]} \quad (11)$$

$f_b^{(1)}$ vanishes in the non-relativistic limit, as is also the case in the kinetic theory treatment starting with the non-relativistic Boltzmann equation and the Maxwell-Boltzmann equilibrium distribution.

3. Relativistic bulk viscosity in Marle's approximation

As mentioned in the introduction, the expression for the stress tensor here used in order to calculate the relativistic bulk viscosity is the one established in Ref.[1] namely,

$$\Pi^{\mu\nu} = m h_\alpha^\mu h_\beta^\nu \mathcal{L}_\eta^\alpha \mathcal{L}_\delta^\beta \int f^{(1)} K^\eta K^\delta d^* K. \quad (12)$$

Here $\mathcal{L}^{\mu\nu}$ is a Lorentz boost with velocity \vec{u} , that is a transformation from an arbitrary reference frame to the frame where the hydrodynamic velocity vanishes. Following the usual procedure, the stress tensor can be written as

$$\Pi^{\mu\nu} = (\Pi^{\mu\nu})^{(a)} + (\Pi^{\mu\nu})^{(s)} + \frac{1}{3} \Pi^{\alpha\beta} \eta_{\alpha\beta} \eta^{\mu\nu} \quad (13)$$

where the first term corresponds to the antisymmetric part, the second to the symmetric part and the third to the trace. It is this third term that will be coupled to the divergence of the hydrodynamic velocity. The proposed constitutive equation is thus

$$\frac{1}{3} \eta_{\nu\mu} \Pi^{\nu\mu} = -\eta \mathcal{U}_{;\mu}^\mu \quad (14)$$

where η is the bulk viscosity. Thus,

$$\eta = \frac{m}{3} \tau \eta_{\nu\mu} \mathcal{O}_a^\mu \mathcal{O}_b^\nu \int f^{(0)} \left\{ \frac{k^2 \gamma_k^2}{3 z c^2} - \gamma_k \left[1 + \frac{k_B}{C_n} \left(1 + \frac{\gamma_k}{z} - \frac{\mathcal{G} \left(\frac{1}{z} \right)}{z} \right) \right] \right\} K^a K^b d^* K \quad (15)$$

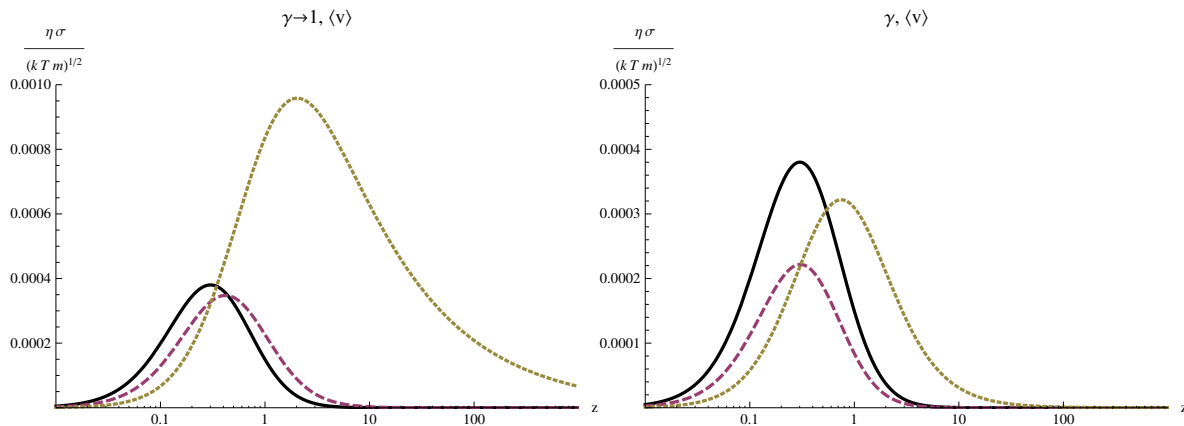


Figure 1. Normalized bulk viscosity with $\mathcal{V} = \langle k \rangle$ (left) and $\mathcal{V} = \langle \gamma_k \rangle \langle k \rangle$ (right). The dotted line shows the result obtained in the present paper while the dashed curve corresponds to value obtained using the method of moments in Ref.[2]. The continuous line corresponds to the value obtained by using the complete kernel in the hard sphere's case [2]. Should be pointed out that in the non-relativistic limit $\langle \gamma_k \rangle \sim 1 + z$.

where the tensor $\mathcal{O}^{\mu\nu} = h^\mu_\alpha \mathcal{L}^{\alpha\nu}$ has the property $\mathcal{O}^{\mu 4} = 0$. Other properties of this tensor can be found in Ref. [1]. Also, since all $a \neq b$ lead to vanishing integrals and changing the variable to γ_k one obtains

$$\eta = 4\pi c^3 \frac{m}{9} \tau \int f^{(0)} \left\{ \frac{k^2 \gamma_k^2}{3z c^2} - \gamma_k \left[1 + \frac{k_B}{C_n} \left(1 + \frac{\gamma_k}{z} - \frac{\mathcal{G}\left(\frac{1}{z}\right)}{z} \right) \right] \right\} \gamma_k^2 k^2 \sqrt{\gamma_k^2 - 1} d\gamma_k. \quad (16)$$

The corresponding integrals can be readily calculated and simplifying one obtains

$$\eta = \frac{1}{3} n m c^2 \tau \frac{[z(2 - 20z^2) \mathcal{G}\left(\frac{1}{z}\right) + 13z^2 \mathcal{G}^2\left(\frac{1}{z}\right) - 2z \mathcal{G}^3\left(\frac{1}{z}\right) - 3z^2]}{3 \left[1 + 5z \mathcal{G}\left(\frac{1}{z}\right) - \mathcal{G}^2\left(\frac{1}{z}\right) - z^2 \right]} \quad (17)$$

which in the low z , non-relativistic limit, vanishes:

$$\eta \approx \frac{5}{18} n m c^2 \tau z^3. \quad (18)$$

In Figs. (1) and (2) the results here obtained are shown together with those obtained by Cercignani and Kremer in Ref.[2] using the complete collision kernel with hard spheres' model and Marle's approximation with the method of moments. The relaxation time parameter used for the plots is calculated as:

$$\tau = \frac{1}{4n\pi\sigma\mathcal{V}} \quad (19)$$

where \mathcal{V} is a characteristic speed given by either the sound speed v_s , a mean velocity $\langle k \rangle$ or either of those values multiplied by an adjusting factor $\langle \gamma_k \rangle$ corresponding to Marle's approximation, as broadly discussed in Ref.[3]. The choice of these velocities, as pointed out in Ref. [2], arises from the fact that both v_s and $\langle k \rangle$ are of the same order as the average of the relative velocity, which is used to calculate τ in the non-relativistic case.

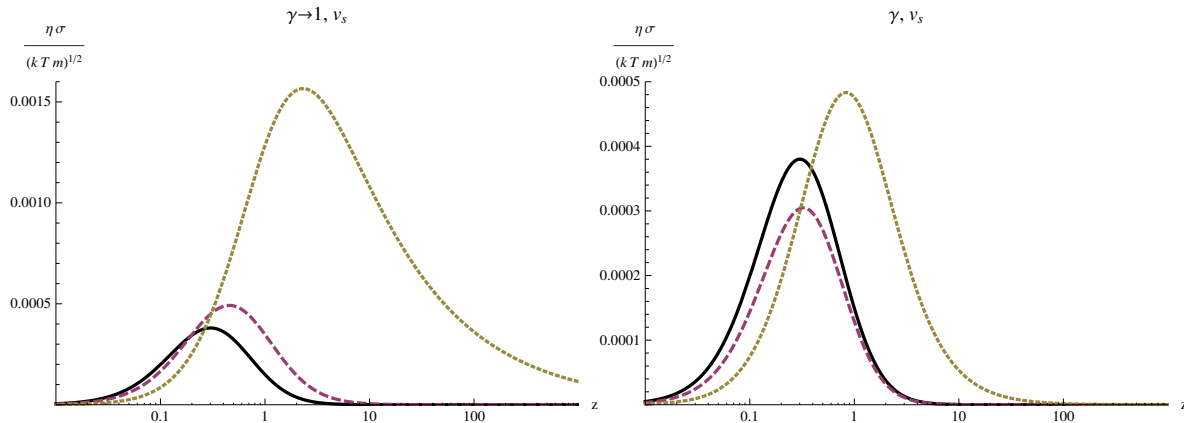


Figure 2. Normalized bulk viscosity with $\mathcal{V} = v_s$ (left) and $\mathcal{V} = \langle \gamma_k \rangle v_s$ (right). The dotted line shows the result obtained in the present paper while the dashed curve corresponds to value obtained using the method of moments in Ref.[2]. The continuous line corresponds to the value obtained by using the complete kernel in the hard sphere's case [2].

4. Final Remarks

Viscous dissipation in non-relativistic simple fluids is only due to shear stresses. As temperature increases relativistic effects in molecular dynamics lead to a non-vanishing bulk viscosity which increases dissipation. This effect has important consequences on cosmological fluid dynamics since bulk viscosity is a source of entropy production in homogeneous and isotropic space-times [10].

In this work we obtained the bulk viscosity coefficient as a function of the relativistic parameter z using a relaxation time approximation, namely Marle's model. A similar calculation is carried out by Cercignani & Kremer in Ref.[2] using the method of moments. The novel feature of this work is the use of the standard Chapman-Enskog procedure and the concept of chaotic velocity, included both in the deviation from equilibrium of the distribution function and the expression for the stress-energy tensor in terms of Lorentz transformations. It should be emphasized that the relaxation equation is here used directly without any additional iterative procedures or corrections. The result obtained features good agreement when the correcting factor introduced in Ref.[3] for the characteristic time is considered. Moreover, the consideration of the $\langle \gamma_k \rangle$ factor in the expression for τ improves the behavior of the transport coefficient obtained here within Marle's model as well as the one calculated in Ref.[2].

The behavior of the bulk viscosity is shown in Figs. (1) and (2). The coefficient vanishes in the low temperature limit, grows with temperature and decreases after reaching a peak value for thermal energies comparable to the particles' rest mass energy. In the ultra-relativistic limit, the coefficient asymptotically approaches zero. The physics behind this behavior is an interesting subject and will be addressed elsewhere.

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