On a parity property in the thermal Sunyaev-Zel'dovich effect

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Abstract

The main issue in this paper is to discuss a parity property that appears in the expressions for the distorted spectrum of the thermal Sunyaev-Zel'dovich effect. When using the convolution integrals method involving scattering laws we argue that the distorted spectrum contains a new term, which is an odd power of the frequency. Such a term, absent in the conventional approaches, implies a crossover frequency which differs in value form the ones reported in the literature by a significant, in principle observable, amount. Also, such term casts doubt on the demanding need of computing complicated relativistic calculations. The relationship of our approach with the existing calculations is discussed.

1 Introduction

The Sunyaev-Zel'dovich (SZ) effect is a signature left in the cosmic microwave background (CMB) by collapsed structures [1] [2]. The thermal Sunyaev-Zeldovich effect arises from the frequency shift of CMB photons that are scattered by the hot electrons contained in structures such as galaxy clusters. The frequency dependence of this effect results, for a given line of sight, in an intensity decrease in the Rayleigh-Jeans region of the CMB spectrum and to an intensity increase at Wien's region. The effect has been detected in observations of some rich, X-ray luminous clusters. Besides this overall effect in the light spectrum, of most interest is the value of the crossover frequency, which is an observable quantity commonly used in the determination of cosmological parameters [3] [4]. This observable is a strong ingredient in our foregoing discussion.

The study of the non-relativistic and relativistic thermal effect using analytical methods has led many authors to agree, one way or another, upon the fact that the distorted spectra arising form Compton scattering can be expressed as a power series containing terms which are even powers of the frequency ν, except for a first order in ν that is present due to the diffusion approximation [5] [6] [7] [8]. The parameter $z = \frac{kT_e}{mc^2}$ used as a discriminant to indicate when the relativistic corrections are important is usually kept up to second order effects. In this paper, we challenge these results. We claim that if the correct basic physics behind the inverse Compton scattering is used, a term which is odd in ν, in fact of order $z^2\nu^3$ must appear, leading to a couple of singular results. The first one is that the crossover frequency, ν_c, is subject to a 2.8 percent correction with respect to the value predicted by diffusion, in comparison to the 1.19 percent correction obtained with the relativistic equation. These values are taken for $kT_e = 5KeV$. This could be presumably tested with precision observations. The second implication is that, even when $kT_e = 15KeV$, this odd power term, not only favors a value of ν_c with an 8.4 correction with respect to the 4.3 percent predicted by relativistic corrections, but clearly indicates that the curves for the distorted spectra are practically indistinguishable from those obtained by numerical methods, specially in Wien's region. Moreover, it is precisely in this limit where claims have been made pointing out the opposite [9]. In short, we sustain the old claim issued by Sunyaev 25 years ago [10], stating that there is no need to perform relativistic corrections to obtain the correct spectra even at the upmost values for $kT_e = 15KeV$.

These arguments will be subsequently developed in sections two and three of this paper.

2 Gaussian scattering laws

We begin this paper by recalling the approach followed previously to discuss the SZ effect in terms of a scattering law [10] [11]. This method is based upon the fact that the distorted radiation spectrum which originates from inverse Compton scattering between electrons and photons in a certain cluster is given by:

$$I(\nu) = \int_0^\infty I_o(\bar{\nu}) G_s(\bar{\nu}, \nu) d\bar{\nu} \tag{1}$$

Here, $I(\nu)$ is the scattered radiation off the plasma, $I_o(\nu) = \frac{2h\nu^3}{c^2}(exp(\frac{h\nu}{kT_R}) - 1)^{-1}$ the undistorted spectrum, where T_R is the CMB temperature, and $G_s(\bar{\nu}, \nu)$ the scattering law. For the thermal non-relativistic effect we have argued that the form $G_s(\bar{\nu}, \nu)$ is given by [10]

$$G_s(\bar{\nu}, \nu) = (1 - \tau)\delta(\bar{\nu} - \nu) + \tau G(\bar{\nu}, \nu) \tag{2}$$

where τ is the optical depth for a thin plasma, and

$$G(\bar{\nu}, \nu) = \frac{1}{\sqrt{2\pi}\sigma(\nu)} e^{-(\frac{\bar{\nu} - (1-2z)\nu}{\sqrt{2}\sigma(\nu)})^2}. \tag{3}$$

Here, $\sigma^2(\nu) = 2\frac{kT_e}{m_e c^2}\nu^2 = 2z\nu^2$ is the square of the width of the spectral line at frequency ν, and m_e and T_e are the mass and temperature of an electron and the electron gas, respectively.

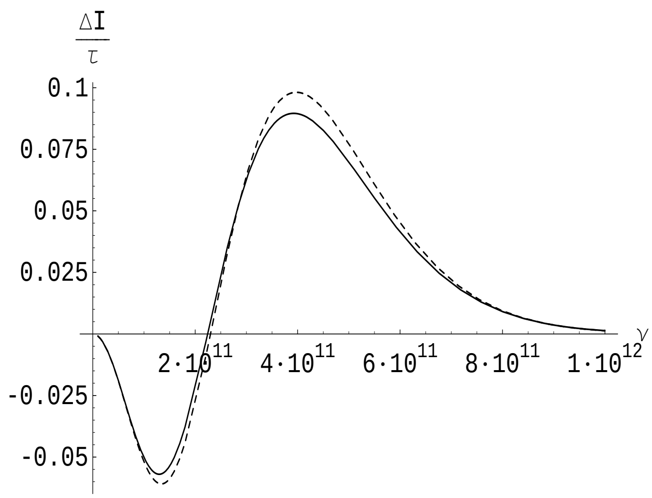

Figure 1: CMB distortion for the case of Eq. (4) (dashed line) compared with the relativistic curve, Eq. (5) (solid line), here $kT_e = 7.5KeV$. Both curves are practically identical deep in the Wien region. The frequency is given in Hz and $\frac{\Delta I}{\tau}$ is given in units of $\frac{2(kT_R)^3}{(hc)^2}$.

The first term in Eq. (1) is self explanatory, it represents the probability that a photon traverses the plasma unscattered. The second term is a modified Gaussian probability function describing the scattering of a photon with an incoming frequency $\bar{\nu}$ by an electron whose average kinetic energy is $\frac{1}{2}kT_e$. The reason of why the outgoing frequency is shifted by a factor $2z\nu$ arises precisely form the fact that the average frequency of a photon scattered by electrons with speeds $u = \beta c$ will exhibit a temperature blue shift given by $2z\nu$ [10] [11].

Inserting Eqs. (2) and (3) into (1), and upon integration [11], one is lead to the result that

$$\frac{\Delta I(\nu)}{\tau} = -2z\nu\frac{\partial I_o}{\partial \nu} + z\nu^2\frac{\partial^2 I_o}{\partial \nu^2} + 2z^2\nu^2\frac{\partial^2 I_o}{\partial \nu^2} - 2z^2\nu^3\frac{\partial^3 I_o}{\partial \nu^3} + \frac{z^2\nu^4}{2}\frac{\partial^4 I_o}{\partial \nu^4} \qquad (4)$$

to second order in z and first order in τ. Eq. (4) is a useful expression in order to discuss the SZ thermal effect. Nevertheless, several characteristics of this result must be underlined. Here, of course, $\Delta I(\nu) = I(\nu) - I_o(\nu)$.

Setting aside the fact that to first order in z Eq. (4) is identical to the diffusion approximation described by the Kompaneets equation [1] [2] [12], a fact that has been analyzed in Ref. [13], we have other interesting features. The first one is the presence of a third order derivative term in the right hand side of Eq. (4) and the other one is that the last term leads to the somewhat surprising fact that the Wien side of the distortion curve is practically identical to its relativistic counterpart [11]. This is clearly exhibited in Fig.(1) for $kT_e = 7.5KeV$.

A careful examination of the analytical expression for the relativistic thermal SZ effect derived by Sazonov and Sunyaev shows the absence of a third order derivative term. Indeed, Eq. (12) of Ref. [5] for a static cluster ($V = 0$) is identical to the expression:

$$\begin{aligned}\frac{\Delta I_{SS}}{\tau} &= (-2\,z + \tfrac{17}{5}z^2)\,\nu\frac{\partial I_o}{\partial \nu} + (\,z - \tfrac{17}{10}z^2)\,\nu^2\frac{\partial^2 I_o}{\partial \nu^2} \\ &\quad + \tfrac{7}{10}z^2\nu^4\frac{\partial^4 I_o}{\partial \nu^4}\end{aligned} \qquad (5)$$

a fact that can be easily proved by cumbersome but straightforward algebra. Before attempting to analyze if a third order derivative term should arise in the distortion curve, let us go back one step and ask how is it possible to obtain an equation for $\frac{\Delta I(\nu)}{\tau}$ without the third order derivative term. We don't question the first derivative term, since it appears already in the diffusion approximation. The non-existence of the ν^3 term can be achieved by writing the scattering law for G_s as follows:

$$G_s = (1-\tau)\delta(\bar{\nu} - (1-2z\tau)\nu) + \tau G'(\bar{\nu}, \nu) \qquad (6)$$

where

$$G'(\bar{\nu}, \nu) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(\frac{\bar{\nu}-\nu}{\sqrt{2}\sigma})^2} \qquad (7)$$

Physically, this implies that $G'(\bar{\nu}, \nu)$ is a purely Gaussian function peaked at $\bar{\nu} = \nu$ [14] and that the central limit theorem governs the scattering of photons by electrons [15]. Nevertheless, the equation would also imply a blue shift in the frequency for the unscattered photons. In fact, any even function $G'(\alpha)$ in the variable $\alpha = \frac{\bar{\nu}-\nu}{\sqrt{2}\sigma}$ would suppress the odd order higher derivative terms, as in Eq. (6). Although we believe that this is physically incorrect, substitution of Eqs. (6) and (7) into Eq. (1) leads to an interesting result, namely,

$$\frac{\Delta I(\nu)_a}{\tau} = -2z\nu\frac{\partial I_o}{\partial \nu} + z\nu^2\frac{\partial^2 I_o}{\partial \nu^2} + \frac{1}{2}z^2\nu^4\frac{\partial^4 I_o}{\partial \nu^4} \qquad (8)$$

In Fig. (2) we exhibit the effects of this result compared with the Kompaneets approximation and with the full relativistic curve of Sazonov and Sunyaev , Eq. (5) [5] for $kT = 10KeV$. In Fig.(3) the curve arising form Eq. (4) is included. The reader may wonder why all this fuss about the higher order in z corrections but we believe that the underlying physics is basic for understanding the effect and, as we shall see below, also to grasp the nature of the relativistic corrections to G_s.

Thus, we are brought back to the original question, namely, if there is a third order derivative present in Eq. (4). To obtain a guide for replying we went back to a detailed examination of two papers, one by Stebbins written six years ago [6] and the publication of Sazonov and Sunyaev on relativistic corrections to both the thermal and kinematic SZ effects [5]. Although based on different approaches, both led to results which turn out to be identical for the thermal SZ effect. The same parity properties were found in related works [6] - [8]. Indeed, Eq.(10) of Ref. [6] is essentially an expression for the distorted spectra written in terms of the occupation number $n_\gamma = (e^{\frac{h\nu}{kT_R}} - 1)^{-1}$ instead of the intensity $I(\nu)$, noticing that

$$\frac{\Delta n}{n_o\tau} = \frac{\Delta I}{I_o\tau} \qquad (9)$$

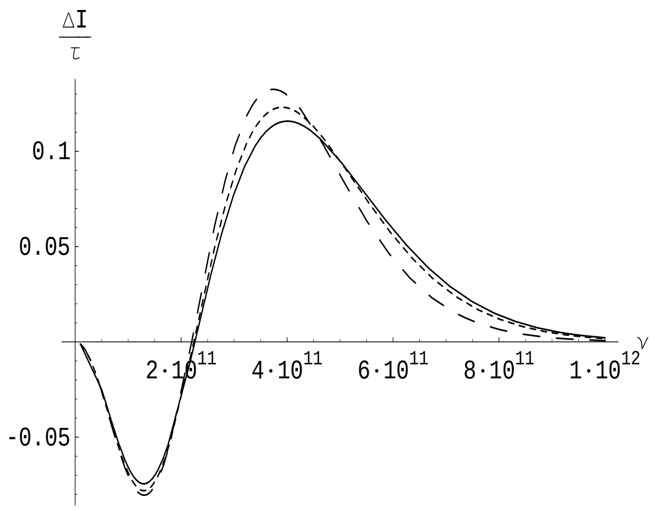

Figure 2: CMB distortion for the case of Eq. (8) (short dashed line) compared with the Kompaneets approximation (long dashed line) and the relativistic curve, Eq. (5) (solid line), $kT_e = 10KeV$. The order z^2 curves are practically identical deep in Wien's region. The frequency is given in Hz and $\frac{\Delta I}{\tau}$ is given in units of $\frac{2(kT_R)^3}{(hc)^2}$.

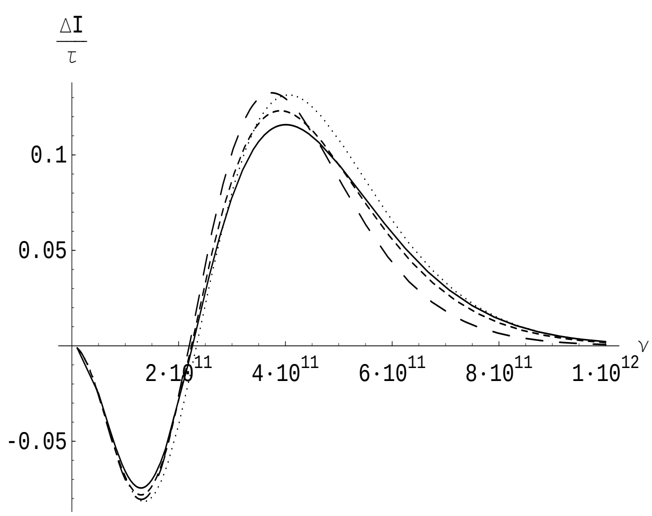

Figure 3: The same as is Fig. (2) now including a comparison with Eq. (4), (dotted line). The third order derivative term induces an increase in the crossover frequency and a *better* agreement with the relativistic curve at high frequencies.

where τ is the optical depth, identifying $\epsilon = h\nu$ and assuming that $h\nu \ll mc^2$ in that equation. The expansion of all terms of Eq.(10) of Ref. [6] is a laborious but straightforward calculation that shows that the third order derivative in I_o disappears and that the resulting expression is identical to Eq.(5). The surprising result of the comparison between the non-relativistic limit of Eq.(5) and Eq. (4) is that the third order derivative in I_o ought to be absent. In our context, this would imply that, for low temperature clusters $kT_e \simeq 5KeV$, Eq. (8) should be essentially correct, which we object because it is physically questionable.

This leads us to a different test, namely the value of the critical frequency ν_c defined by $\Delta I(\nu_c) = 0$. One gets, by the Kompaneets equation that $\nu_c = 220.027$ GHz, a value which according to the relativistic correction increases, for $kT_e = 5KeV$, only to 222.67 GHz, a 1.19 percent correction. However, if the third order derivative is present in Eq.(4), as we assert, $\nu_c = 226.39$ GHz, giving a more significant correction of 2.81 percent. The simple physics behind Eq.(4) is quite attractive but merely theoretical up to this point. Notice also that from the observation of the results, it is really hard to see which is in agreement with observations, all follow the same pattern, specially in Wien's region.

Finally, a comment on the last term of Eq.(4). Our method, which is non-relativistic, yields a numerical factor of $\frac{5}{10}$ instead of the $\frac{7}{10}$ factor obtained in references [5] and [6] using explicit relativistic corrections. The question is if this difference and possible improvements can be obtained using the scattering law approach in the relativistic case. We turn to this question in the following section.

3 Relativistic scattering law

From the discussion of the previous section it is rather clear that in order to introduce relativistic corrections into the scattering Kernel $G_s(\bar{\nu}, \nu)$ two facts must be considered. First, the energy of an electron has to be written in its relativistic form $E = mc^2(\gamma - 1)$ and second, their velocity distribution is no longer given by Maxwell's distribution, but by its relativistic extension. Nevertheless, since the relativistic parameter z is small even at high temperature clusters, $kT_e = 15.1KeV$, we may simply generalize the scattering law, Eqs. (2) - (3), allowing for z^2 corrections for the peak shift and $\sigma(\nu)$, namely

$$G_r(\bar{\nu}, \nu) = (1 - \tau)\delta(\bar{\nu} - \nu) + \frac{1}{\sqrt{2}N\sigma(\nu)} H(\frac{\bar{\nu} - (1 - \varepsilon)\nu}{\sqrt{2}\sigma(\nu)}) \tag{10}$$

In this case, the evaluation of the expression $I(\nu) = \int_0^\infty I_o(\nu) G_r(\bar{\nu}, \nu) d\bar{\nu}$ leads to a generic Kompaneets type equation. This is achieved performing the change of variable:

$$\alpha = \frac{\bar{\nu} - (1 - \varepsilon)\nu}{\sqrt{2}\sigma(\nu)} \tag{11}$$

with normalization factor

$$N = \int_{-\infty}^{\infty} H(\alpha) d\alpha \tag{12}$$

and identifying

$$\Delta\nu(\alpha) = \sqrt{2}\sigma(\nu)\alpha - \epsilon\nu \tag{13}$$

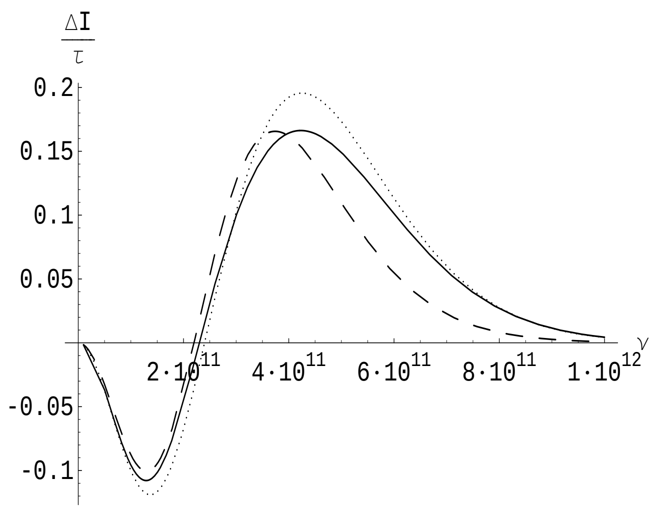

Figure 4: CMB distortion for the case of Eq. (4) (solid line) compared with Eq. (5) (short dashed line) and with the Kompaneets approximation (long dashed line) , $kT_e = 15.1KeV$. The crossover frequency would have a large increase due to the presence of the third order derivative term. The frequency is given in Hz and $\frac{\Delta I}{\tau}$ is given in units of $\frac{2(kT_R)^3}{(hc)^2}$.

The resulting distortion curve reads:

$$\begin{array}{l} \frac{\Delta I_r}{\tau} = -\epsilon\nu\frac{\partial I_o}{\partial\nu} + (\frac{\epsilon^2}{2})\,\nu^2\frac{\partial^2 I_o}{\partial\nu^2} + (\frac{\sigma^2}{N})\frac{\partial^2 I_o}{\partial\nu^2}\int_{-\infty}^{\infty}\alpha^2 H(\alpha)d\alpha \\ +(\frac{\sigma^2\epsilon\nu}{N})\frac{\partial^3 I_o}{\partial\nu^3}\int_{-\infty}^{\infty}\alpha^2 H(\alpha)d\alpha + (\frac{\sigma^4}{6N})\frac{\partial^4 I_o}{\partial\nu^4}\int_{-\infty}^{\infty}\alpha^4 H(\alpha)d\alpha \end{array} \tag{14}$$

Eq. (4) is a particular case of Eq. (14) for $\epsilon = -2z$, $\sigma^2(\nu) = 2z\nu^2$ making $H(\bar{\nu},\nu) = \frac{1}{\sqrt{2\pi}\sigma(\nu)}e^{-(\frac{\bar{\nu}-(1-2z)\nu}{\sqrt{2}\sigma(\nu)})^2}$. Of course, the series can be trivially taken to arbitrary order in z^n. It is interesting to notice that the coefficient of the third order derivative term in Eq.14) is independent of relativistic effects up to second order in z and would yield $-2z\nu^3$ even taking into account relativistic corrections. In Fig. (4) we exhibit the effects of Eq. (4) compared with the Kompaneets approximation and with the full relativistic curve of Sazonov and Sunyaev [5] for $kT = 15.1KeV$. The critical frequency shift correction will now increase from 4.347 to 8.414 percent, basically due to the presence of the third derivative term.

4 Discussion

The present paper not only shows that the ν^3 term appearing in Eq. (4) is essentially enough to account for the relativistic features of the SZ thermal effect without entering into long and sophisticated calculations, but provides a bridge between the convolution integral approaches to the thermal SZ effect and the expansions in terms of derivatives of the undistorted number occupation . The calculation of convolution integrals in the logarithmic space, which has been used by other authors [3] [9], prevents the establishment of expressions such as Eq. (14) that may relate the approaches. On the other hand, the calculation of the convolution integrals in the frequency space fills this gap. Similar techniques applied to CMB distortions have been discussed in other contexts [16].

To end this work we wish to underline the fact that the presence of the third derivative term is directly directed to the frequency shift included in the kernel of Eq. (3). If we take into account that, on the average, a scattering produces a slight mean change of photon energy [3] [10], then direct use of the central limit theorem leads to Eq. (4).

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