APPROXIMATE SOLUTIONS FOR DORODNITZYN'S GASEOUS BOUNDARY LAYER LIMIT FORMULA

A PREPRINT

Carla V. Valencia-Negrete Department of Physics and Mathematics
Universidad Iberoamericana Ciudad de México
Prolongación Paseo de la Reforma 880, Mexico City —01219, MEXICO
carla.valencia@ibero.mx

January 4, 2022

ABSTRACT

Oleinik's *no back-flow* condition ensures the existence and uniqueness of solutions for the Prandtl equations in a rectangular domain $R \subset \mathbb{R}^2$. It also allowed us to find a limit formula for Dorodnitzyn's stationary compressible boundary layer with constant total energy on a bounded convex domain in the plane $\mathbb{R}^2$. Under the same assumption, we can give an approximate solution u for the limit formula if $|u| \lll 1$ such that:

$$u(z) \cong \delta * c * \left[z + \frac{6}{25} \cdot \frac{1}{2i_0} \cdot \frac{4U^2}{3} z^4 \right] + o(z^5),$$

that corresponds to an approximate horizontal velocity component when a small parameter ϵ given by the quotient of the maximum height of the domain divided by its length tends to zero. Here, $c > 0$, δ is the boundary layer's height in Dorodnitzyn's coordinates, U is the *free-stream* velocity at the upper boundary of the domain, and T_0 is the absolute surface temperature.

***K*eywords** Boundary layer theory · theory Gas dynamics

1 Introduction

First, the limit formula is rewritten as a non-linear ordinary differential equation of order 1 in Lemma 1,

$$\frac{\partial u}{\partial s} = c \left[1 - \left(u^2/2i_0 \right) \right]^{-6/25},$$

where $i_0 = c_p\, T_0$ the constant total energy value, c_p is *specific heat at constant pressure*, T_0 is the absolute surface temperature, $c = \partial u/\partial s|_{s=0} > 0$ is a strictly positive constant that represents the *no back-flow* condition continuous extension to the lower boundary, and the fractional exponent $-6/25 = 19/25 - 1$ comes from the empirical *Power-Law* $\mu/\mu_h = (T/T_h)^{\frac{19}{25}}$ for the absolute temperature T and the dynamic viscosity μ with the *free-stream dynamic viscosity* $\mu_h > 0$, and the *free-stream temperature* $T_h > 0$ at the upper boundary [3, p. 46].

Then, it is shown in Lemma 2 that if $|u| \lll 1$, the non-linear term can be expressed as a power series $\left[1 - \left(u^2/2i_0 \right) \right]^{-6/25} = 1 + (6/25)(1/2i_0)\, u^2 + o(u^4)$. In Theorem 1, we seek a solution of the form of a fourth degree polynomial, $Az + Bz^2 + Cz^3 + Dz^4$, in the same way it is done in [1, 2]. The existence and unicity of the solutions comes from the Darboux sums limit of the Riemann integral taken to solve Eq. (2).

2 Problem Statement

The limit formula (Eq. (1) in the next page) is presented as a non-linear ordinary differential equation of order 1.

Definition 1. *Let $\tilde{D} \subset \mathbb{R}^3$ be a bounded open subset of the three dimensional Euclidean space such that its projection $\tilde{D} \cap \mathbb{R}^2$ is a planar region D,*

$$D{:}=\left\{(x,y)\in\mathbb{R}^2 \mid 0<x<L\,\&\; 0<y<h(x)\right\},$$

where $L \ggg h > 0$, and $h{:}\,[0,L]\to(0,\infty)\in C^2([0,L])$ has derivatives up to order two in $(0,L)$ with extension to the corresponding extremes, $\{0,L\}$, of this interval, and $h(0)=h(L)=\delta$. Moreover, the upper boundary h has only one critical point c which is a global maximum, $h(c)=H\geq h(x)\;\forall x\in[0,L]$. We denote the topological boundary of D by ∂D.

Definition 2. *Consider: (i) the first coordinate ℓ of Dorodnitzyn's diffeormorphism such that for all $(x,y)\in D$, $\ell\,(x,y)=c_1\,\left[1-\left(U^2/2i_0\right)\right]\,x$, where the* free-stream *velocity $u|_{y=h(x)}= U>0$, $c_1=p_0\,T_0^{2b/b-1}$, $p|_{y=0}=p_0$ is the pressure at the lower boundary, T_0 is absolute surface temperature, and $b=1.405$ is the exponent of the* adiabatic *and* polytropic *atmosphere constant, $p\,V^b=cte.$, for the volume $V=\iiint\hat{D}d\mathbf{x}$ [4, p. 35]; (ii) the second coordinate s of Dorodnitzyn's diffeomorphism*

$$s\,(x,\hat{y})=\int_0^{\hat{y}}\rho(x,y)dy,$$

where ρ is the density; and (iii) the upper boundary in the Dorodnitzyn's domain, $\delta(x,h(x))=\int_0^{h(x)}\rho(x,y)dy$. Then, $\mathbf{s}=(\ell,s)$ is a diffeomorphism defined in the domain D [5].

Lemma 1. *Let D be a convex domain and $\mathbf{s}:D\to\mathbf{s}(D)$ as described in Definition 1 and 2. Assume that (i) $u\in C^2(D)$, (ii) $\partial u/\partial s>0$ in D with a continuous extension to $\partial u/\partial s|_{s=0}= c>0$ to a positive constant c, (iii) $u(\ell,0)=0$ for all $\ell\in[0,1]$, and (iv) the following relation:*

$$f\,\frac{\partial^2 u}{\partial s^2} \quad=\quad \frac{\partial f}{\partial s}\,\frac{\partial u}{\partial s}, \tag{1}$$

where $f=\left[1-\left(u^2/2i_0\right)\right]^{-6/25}>0$, and i_0 is an strictly positive constant. Then, u is a solution of Eq. (1) if and only if it verifies Eq. (2):

$$\frac{\partial u}{\partial s} \quad=\quad c\left[1-\left(u^2/2i_0\right)\right]^{-6/25}. \tag{2}$$

Proof. If $g=\partial u/\partial s$, then Eq. (1) becomes $(\partial g/\partial s)/g=(\partial f/\partial s)/f$. This is, $\int_0^y\frac{\partial}{\partial s}\left[ln\left(g(s)\right)\right]ds=\int_0^y\frac{\partial}{\partial s}\left[ln\left(f(s)\right)\right]ds$. Thus, $exp\left[ln\left(g(y)/g(0)\right)\right]=exp\left[ln\left(f(y)/f(0)\right)\right]$. Therefore, $g(y)=\left(g\left(0\right)/f\left(0\right)\right)f(y)$. The *no-slip* condition $u(x,0)=0$ for all $x\in[0,L]$ implies that $f(0)=1$. Finally, we substitute $g=\partial u/\partial s$, and we obtain Eq. (2). ☐

3 Analytic solutions for the limit formula

In Lemma 2, we show that there is an expression of the non-linear term $\left[1-\left(u^2/2i_0\right)\right]^{-6/25}$ as a convergent power series if $|u|\lll 1$. Finally, following [1, 2], in Theorem 1, we may seek a solution of the form of a fourth degree polynomial with coefficients chosen so that the boundary conditions of the problem are satisfied.

Lemma 2. *Let $|u|\lll 1$. Then,*

$$\left[1-\left(u^2/2i_0\right)\right]^{-6/25}=1+\frac{6}{25}\cdot\frac{u^2}{2i_0}+\frac{6}{25}\cdot\frac{u^4}{2(2i_0)^2}+\frac{1}{2}\cdot\frac{6}{25}\cdot\frac{u^4}{(2i_0)^2}+o(u^6). \tag{3}$$

Proof. If $\sigma=1-(u^2/2i_0)\cong 1$, then $\theta=(-6/25)\ln\left(1-(u^2/2i_0)\right)\lll 1$. We apply the *Maclaurin formula* for the exponential, $e^\theta=1+\theta+(1/2)\theta^2+(1/3!)\theta^3+\cdots$, and the Taylor series expansion for the logarithmic function if $|\sigma|<1$, $log(1+\sigma)=\sigma-(1/2)\sigma^2+(1/3!)\sigma^3+\cdots$, so that:

$$\begin{aligned}\left[1-\left(u^2/2i_0\right)\right]^{-6/25}&=\exp\left[\frac{-6}{25}\,\ln\left(1-\frac{u^2}{2i_0}\right)\right],\\&=\exp\left[\frac{-6}{25}\,\left\{\ln\left(1+\frac{u}{\sqrt{2i_0}}\right)+\ln\left(1-\frac{u}{\sqrt{2i_0}}\right)\right\}\right],\\&=1+\frac{6}{25}\cdot\frac{u^2}{2i_0}+\frac{6}{25}\cdot\frac{u^4}{2(2i_0)^2}+\frac{1}{2}\cdot\frac{6}{25}\cdot\frac{u^4}{(2i_0)^2}+\\&\quad+\frac{6}{25}\cdot\frac{u^6}{3(2i_0)^3}+\frac{6}{25}\cdot\frac{u^6}{2(2i_0)^3}+o(u^8).\end{aligned}$$

□

Remark 1. *In order to identify the coefficients of a polynomial expression for* u*, we observe Dorodnityzn's steps [1, 2]. If we take the upper boundary conditions for the second order derivative from the problem stated by Dorodnitzyn on the domain* $\mathbf{s}(D)$*, then the boundary conditions establish a system of four equations for the coefficients* A*,* B*,* C*,* D *so that* u *can be expressed as a fourth degree polynomial* $Az + Bz^2 + Cz^3 + Dz^4$ *for* $z = s/\delta$ *if* $A = U\,[2 + (\lambda/6)]$*,* $B = -U\,(\lambda/2)$*,* $C = U\,[(\lambda/2) - 2]$*, and* $D = U\,[1 - (\lambda/6)]$*, where* λ *is the Pohlhausen coefficient* $\lambda = \left[-\delta/\left(1 - U^2\right)\right] \partial U/\partial \ell$*, and* $\lambda = 0$ *if* U *is constant.*

Remark 2. *For each fixed value of* $x \in [0, L]$*, the height's normalization* $z(x) = s(x)/\delta(x) \in [0, 1]$*. Additionally,* $\partial u/\partial z = \delta * \partial u/\partial s$ *because* $\partial u/\partial z = (\partial u/\partial s)(\partial s/\partial z) + (\partial u/\partial \ell)(\partial \ell/\partial z)$ *and* ℓ *is independent of* z*.*

Theorem 1. *Under the same conditions of Lemma 1, Eq. (2) has a unique solution*

$$u(z) = \delta * c * \left[z + \frac{6}{25} \cdot \frac{1}{2i_0} \cdot \frac{4U^2}{3} z^4\right] + o(z^5),$$

for the normalized height $z = s/\delta \in [0, 1]$*.*

Proof. Given Eq. (2), we can substitute Eq. (3), so that:

$$\begin{aligned} \frac{\partial u}{\partial s} &= \frac{1}{\delta} \frac{\partial u}{\partial z}, \\ &= c\left[1 - \left(u^2/2i_0\right)\right]^{-6/25}, \\ &= c\left[1 + \frac{6}{25} \cdot \frac{u^2}{2i_0} + \frac{6}{25} \cdot \frac{u^4}{2(2i_0)^2} + \frac{1}{2} \cdot \frac{6}{25} \cdot \frac{u^4}{(2i_0)^2} + o(u^6)\right]. \end{aligned} \tag{4}$$

Because the *stream-flow velocity* U is constant, then the Pohlhausen coefficient $\lambda = 0$. Thus, we can seek a solution with a fourth degree polynomial form $u(z) = 2Uz - 2Uz^3 + Uz^4$. If we replace this expression in the Eq. (4):

$$\begin{aligned} u(z) &= \delta * c * \int_0^z \left[1 + \frac{6}{25} \cdot \frac{u(\tau)^2}{2i_0} + \frac{6}{25} \cdot \frac{u(\tau)^4}{2(2i_0)^2} + \frac{1}{2} \cdot \frac{6}{25} \cdot \frac{u(\tau)^4}{(2i_0)^2} + o(u^6)\right] d\tau, \\ &= \delta * c * \left[z + \frac{6}{25} \cdot \frac{1}{2i_0} \cdot \frac{4U^2}{3} z^4\right] + o(z^5). \end{aligned}$$

□

4 Conclusion

Olga Oleinik's *no back-flow* condition is sufficient to show the existence and unicity of solutions for Dorodnitzyn's gaseous boundary layer limit when the horizontal velocity component is small. The fact that one can find analytic approximations for in a compressible boundary layer when the horizonal component of the velocity is small allows us not only to empirically see that the velocity profile is stable at low velocities, but also to analytically understand that, in this case, one can arrive at flow expressions that linearly depend on the height.

5 Appendix

Funding: This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Declarations of interest: none

References

[1] Dorodnitzyn, A. A. Laminar Boundary Layer in Compressible Fluid. In *Comptes Rendus de l'Académie des Sciences de l'URSS (Doklady)*, **34**:8, pages 213–219, 1942.

[2] Lees, L. Stability of the Supersonic Laminar Boundary Layer With a Pressure Gradient. In *Princeton University NJ James Forrestal Research Center*, 1950-11-20.

[3] Smits, A. J. Dussauge, J.-P. Turbulent Shear Layers in Supersonic Flow. Springer-Verlag, 2006.

[4] Tietjens, O. G. Fundamentals of Hydro- and Aerodynamics. Based on Lectures by L. Prandtl, Ph. D. Dover Publications Inc., 1934.

[5] Valencia-Negrete, C. V., et al. Reynolds' Limit Formula for Dorodnitzyn's Atmospheric Boundary Layer In Convective Conditions. In *International Journal of Applied Mathematics*, Vol. 31, No 4 (2018), pp. 673–695, DOI: 10.12732/ijam.v31i4.12