Synchronization Preservation of Dynamical Networks

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Abstract

Within the context of dynamical systems and statistical mechanics we present the following work. A particularly interesting problem is how a given collective dynamics (for instance, a synchronous motion) can be preserved when important changes occur in the complex dynamical system. Recently, it has been demonstrated that many large-scale complex dynamical networks display a collective synchronization motion. We also know that most of the real world networks are not stationary, in the sense that they are growing, with new nodes continuously being added to the graph (WWW, Internet, Science Citation index, regulatory networks, are just a few of such examples). Therefore it comes out a natural question on how these networks can preserve a given collective dynamics or functioning, while the process of their growth is taking place in time. We propose to develop a methodology to extend some classic results of the dynamical systems theory, preserving the structure of the signs of the real parts of the eigenvalues of the Jacobian matrix in its equilibrium points of the dynamical system. In particular, we present some extensions of the stable manifold theorem and the center manifold theorem. The developed methodology is used to study the problem of preservation of synchronization in chaotic dynamical systems, in particular in the case of dynamical networks. In this work using matrix theory tools, specifically, the methods for triangularization of matrices, the Kronecker product, the multiplicative group structure for triangular matrices, the closure under product and sum of positive defined triangular matrices and the eigenvalues sign-preservation for triangular matrices.The results on preservation of stability and synchronization based on the extensions of the stable manifold theorem and the center manifold theorem show that stability and synchronization can be preserved by transforming the linear part of the synchronization system which, in dynamical networks, is related to the connectivity of nodes. The transformation is performed in time domain. Thus, the results can be also used in the chaos suppression problem. The results include very relaxed conditions to preserve the stability and synchronization. An example is included to illustrate the results.

1 Introduction

A complex network is a large set of interconnected nodes, in which a node is a fundamental unit, that can have different meanings in different situations, such as chemical substrates, microprocessors, computers, schools, companies, papers, webs, people, and so on [1, 3, 6–8, 12, 15, 19, 20, 23–25]. The theory of complex networks seems to offer an appropriate framework for such a large-scale analysis in a representative class of complex systems, with examples ranging from cell biology and epidemiology to the Internet [1,6,23]. These large-scale complex networks often display better cooperative or synchronous behaviors among their constituents.

The complex networks were studied by graph theory, where a complex network is described by a random graph, for which the basic theory was introduced in [11]. Recently, in [24], it is introduced the concept of small-world networks to describe a transition from a regular lattice to a random graph. These networks exhibit a high degree of clustering as in the regular networks and a small average distance between two nodes as in the random networks. Moreover, the random graph model and the small-world networks model are both homogeneous in nature. However, in [6], empirical results show that many large-scale complex networks are scale-free, such as the Internet, the WWW, and metabolic networks, among others. Notably, a scale-free network is inhomogeneous in nature, i.e., most nodes have very few connections but a small number of particular nodes have many connections.

Several feedback schemes for the chaos suppression and synchronization have been widely studied in last two decades (see the reviews in [3, 7, 8, 19, 20, 25]). More recently, the complex networks have opened new challenges for the stabilization of the chaotic dynamics [8]. In this direction, a new problem is to study the conditions under which the stabilization is kept intact during transformation induced by the (feedback or feedforward) interconnections of dynamical systems in a network; i.e., the stability preservation of complex networks. The study of the stability preservation makes sense in the chaos control problems. As matter of fact, the generalized synchronization can be derived even for different systems by finding a diffeomorphic transformation such that the states of the slave system can be written as a function of the states of the master dynamics (see [12] and references therein).

Recently in [4] it is shown that commonly used statistical properties—including the degree distribution, degree homogeneity, average degree, average distance, degree correlation, and clustering coefficient—can fail to characterize the synchronizability of networks. In general, the eigenvalues are among the intrinsic network features which determine the dynamics and which are not derivable from the statistical characteristics. Therefore in the present work, we adopt an spectral approach for analysis of synchronization of complex networks, and we restrict our attention to an particular class of complex networks, modeled by autonomous nonlinear dynamical systems, where the linear part of the vector field is related to the connectivity of the nodes.

The study of the preservation of stability in complex network and nonlinear autonomous dynamical system is not new, for example is well-known that using a change of variables between systems, i.e., using a diffeomorphism in the neighborhood of the origin, the first system is either stable, asymptotically stable or unstable if and only if the second system (the transformed system) is either stable, asymptotically stable or unstable, respectively.Similar results are obtained by computing the multiplication of the vector field in the nonlinear dynamical system by a continuously differentiable function which is positive at the origin [17]. In the case of linear dynamical system there exist several results of preservation of stability, for instance in [10, 14, 22] asymptotically stability is preserved using transformations on rational functions in the frequency domain. Some of these transformations can be interpreted as a special class of noise present in the system also as perturbation on the value of the physical parameters involved in the description of the model. However, the problem of preserving stability by only transforming the Jacobian matrix evaluated at the origin, with minimal restrictions about the nonlinear part of the vector field in nonlinear autonomous systems, has not been studied in depth.

Nevertheless, the problem of the stability preservation has not been addressed for the case of a nonlinear systems with chaotic dynamics, with the unique exception of [13] where the problem, particularly interesting, of how a given collective dynamics (for instance, a synchronous motion) can be preserved when important changes occur in the dynamical system, has been studied. This issue is important for the case of networking systems. We know that most of real world networks are not stationary, in the sense that they are growing, with new nodes continuously being added to the graph (WWW, Internet, Science Citation index, regulatory networks, are just a few of such examples). Therefore it comes out a natural question on how these networks can preserve a given collective dynamics or functioning, while the process of their growth is taking place in time. The stability preservation is studied in [13] for the chaotic synchronization problem. The results show that stability can be preserved by transforming the linear part of the synchronization system which, in complex networks, is related to the connectivity of nodes. Thus, the results can also be used in the chaos suppression problem. This work is inspired by the same objective, therefore two extensions to the Proposition 3 in [13] are presented: an extension of the local stable-unstable manifold theorem and an extension of the center manifold theorem based in the preservation of the signature of the linear part of the vector fields in nonlinear dynamical systems. The results are applied to the chaotic synchronization problem. As we shall see, the results depart from the hypothesis of the existence of a constant state feedback as nominal synchronizing force.

In this chapter, for a first approximation to the problem of preservation of synchronization, our attention is restricted only to uncoupled networks, i.e., we work with autonomous nonlinear dynamical systems. We will give some necessary antecedents so that the reader can understand with facility the results. Thus the Chapter is organized as follows. In section 2, we present some definitions and results about upper triangular matrices, positive definite matrices and Kronecker product of matrices. In section 3, the Fundamental Theorem for linear systems, the local Stable-Unstable Manifold Theorem, the Center Manifold Theorem, The Hartman-Grobman Theorem and the concept of group action are introduced. In section 4 the main result is presented as a generalization of Proposition 3 in [13]. In section 5 we will show that it is possible to preserve synchronization under a class of transformations defined in section 5 and an method for that issue is presented. The numerical experiments on the stability preservation for chaotic synchronization are shown in Section 6. Finally, a conclusion is given in Section 7.

2 Preliminaries results on matrices

In this section, we present some definitions and necessary results about groups and properties of upper triangular matrices, positive definite matrices and specific properties of Kronecker product of matrices.

Definition 2.1. Let λ be an eigenvalue of the $n \times n$ matrix A of multiplicity $m \leq n$. Then for $k = 1, \dots, m$, any nonzero solution w of

$$(A - \lambda I)^k \, w = 0$$

is called a *generalized eigenvector* of A.

Let $w_j = u_j + v_j$ be a generalized eigenvector of the matrix A corresponding to an eigenvalue $\lambda_j = a_j + ib_j$ (note that if $b_j = 0$ then $v_j = 0$). And let

$$B = \{u_1, v_1, \dots, u_k, v_k, \dots, u_m, v_m\}$$

be a basis of $\mathbb{R}^n$ (with $n = 2m - k$ as established by Theorem 1.7.1 and Theorem 1.7.2 in [21]).

Definition 2.2. We define the *stable, unstable and center subspaces*, E^s, E^u and E^c respectively, of a linear system $\dot{x} = Ax$, as

$$\begin{aligned} E^s &= \operatorname{span}\{u_j, v_j \,|a_j < 0\} \\ E^c &= \operatorname{span}\{u_j, v_j \,|a_j = 0\} \\ E^s &= \operatorname{span}\{u_j, v_j \,|a_j > 0\}\,; \end{aligned}$$

that is, E^s, E^u and E^c are the subspaces of $\mathbb{R}^n$ spanned by the real and imaginary parts of the generalized eigenvectors w_j corresponding to eigenvalues λ_j with negative, zero and positive real parts respectively.

Now consider the upper triangular matrices with complex or real elements of the form

$$\begin{bmatrix} \lambda_1 & * & \dots & * \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues. With the multiplication (or usual product) of matrices given by

$$\begin{bmatrix} \mu_1 & * & \ldots & * \\ 0 & \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & \mu_n \end{bmatrix} \begin{bmatrix} \lambda_1 & * & \ldots & * \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix}$$
$$= \begin{bmatrix} \mu_1\lambda_1 & * & \ldots & * \\ 0 & \mu_2\lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & \mu_n\lambda_n \end{bmatrix}$$

and the addition of matrices given by

$$\begin{bmatrix} \mu_1 & * & \ldots & * \\ 0 & \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & \mu_n \end{bmatrix} + \begin{bmatrix} \lambda_1 & * & \ldots & * \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix}$$
$$= \begin{bmatrix} \mu_1+\lambda_1 & * & \ldots & * \\ 0 & \mu_2+\lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & \mu_n+\lambda_n \end{bmatrix}.$$

We denote by Δ_n the set of all upper triangular matrices of $n \times n$ with real or complex elements. It is well-known that this set, Δ_n, is a non-commutative group with the multiplication of matrices, and it is a commutative group with the addition of matrices. Also, the distributive property is valid in this set, but Δ_n is not a field as the real or complex numbers. Notice that the eigenvalues of the product of upper triangular matrices are the product of eigenvalues of both upper triangular matrices, similarly for the addition of upper triangular matrices, their eigenvalues are the addition of eigenvalues from both upper triangular matrices.

Notice that the subset $\Delta_{n_{pd}} \subset \Delta_n$ with all diagonal elements being strictly positive is a multiplicative subgroup of Δ_n. The subset $\Delta_{n_{pd}}$ is closed with respect to the addition of matrices and then a semigroup, i.e., the addition operation is associative, without neutral element or identity, and in general for an element $M \in \Delta_{n_{pd}}$ there does not exist the inverse element $M^{-1} \in \Delta_{n_{pd}}$.

The following set play an important roll in this work:

$$\Lambda_N = \left\{ M \in \mathbb{R}^{n\times n} : M = NT_M N^{-1} \text{ with } T_M \in \Delta_{n_{pd}} \right\}$$

where $N \in \mathbb{R}^{n\times n}$ is a nonsingular fixed matrix.

Notice that Λ_N is a non-commutative group with the multiplication of matrices, and it is a semigroup with the addition of matrices. Again, the distributive property is valid in this set, but Λ_N is not a field as the real or complex numbers.

The following result is known as **Schur's Unitary Triangularization Theorem**.

Lemma 2.1 ([18]). *Given an n by n matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$ in any prescribed order, there is an unitary n by n matrix U such that*

$$T_A = UAU^\top$$

with T_A an upper triangular matrix and the diagonal elements are the eigenvalues of A, i.e., $t_{ii} = \lambda_i$. Furthermore, if the entries of A and its eigenvalues are all real, U may be chosen to be real orthogonal, i.e., $U \in \mathbb{R}^{n\times n}$ such that $UU^\top = U^\top U = I$.

Due to Lemma 2.1, any matrix A of order $n \times n$ is similar to one upper triangular matrix T_A via an unitary matrix U, i.e., any n by n matrix A is triangularizable.

Definition 2.3. If

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix},$$

then the (right) *Kronecker product* of matrices A and B, written $A \otimes B$, is defined to be the partitioned matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{pmatrix},$$

this is a $mn \times mn$ matrix.

In the following result, we state the main properties of the Kronecker product of A and B, written $A \otimes B$.

Lemma 2.2 ([18]). *If the orders of the matrices involved are such that all the operations below are defined, then*

1. *For a real or complex number σ, $(\sigma A) \otimes B = A \otimes (\sigma B) = \sigma (A \otimes B)$;*
2. $(A + B) \otimes C = A \otimes C + B \otimes C$*;*
3. $A \otimes (B + C) = A \otimes B + A \otimes C$*;*
4. $A \otimes (B \otimes C) = (A \otimes B) \otimes C$*;*

5. $(A \otimes B)(C \otimes D) = (AC \otimes BD)$;

6. $A \otimes B = (A \otimes I_n)(I_m \otimes B) = (I_m \otimes B)(A \otimes I_n)$ *where* I_n *is the identity matrix of* $n \times n$ *and similarly for* I_m;

7. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;

8. *The eigenvalues of* $A \otimes B$ *are the* mn *numbers* $\lambda_i \mu_j$ *where* λ_i, $i = 1, 2, \ldots, m$ *are the eigenvalues of* A *and* μ_j, $j = 1, 2, \ldots, n$ *are the eigenvalues of* B.

A consequence of these properties is that the Kronecker product of upper triangular matrices is again an upper triangular matrix.

3 Preliminaries results of dynamical systems

In this section, basic and classical results on properties of dynamical systems are introduced. Specifically, the Fundamental Theorem for linear systems, the local Stable-Unstable Manifold Theorem, the Center Manifold Theorem, The Hartman-Grobman Theorem and the concept of group action are introduced like antecedent for the next section.

First, we present the Fundamental Theorem for linear systems in an informal form.

Let A be an $n \times n$ matrix. The fundamental Theorem establishes that for $x_0 \in \mathbb{R}^n$ the initial value problem

$$\dot{x} = Ax, \quad x(0) = x_0$$

has unique solution for all $t \in \mathbb{R}$ which is given by

$$x(t) = e^{At} x_0.$$

The mapping $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ is called *the flow* of the linear system.

Definition 3.1. If all eigenvalues of the $n \times n$ matrix A have nonzero real part, then the flow, i.e., e^{At}, is called a hyperbolic flow and $\dot{x} = Ax$ is called a hyperbolic linear system.

Definition 3.2. A subspace $E \subset \mathbb{R}^n$ is said to be invariant with respect to the flow $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ if $e^{At} E \subset E$ for all $t \in \mathbb{R}$.

The following is a key technical lemma that will be use through out this work.

Lemma 3.1. *Let* $A \in \mathbb{R}^{n \times n}$. *Then*

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c$$

where E^s, E^u *and* E^c *are the stable, unstable and center subspaces of the linear system, respectively; furthermore,* E^s, E^u *and* E^c *are invariant with respect to the flow* e^{At} *respectively.*

Definition 3.3. Let E be an open subset of $\mathbb{R}^n$ and let $f \in C^1(E)$, i.e., f is a continuous differentiable function defined on E. For $x_0 \in E$, let $\phi(t, x_0)$ be the solution of the initial value problem

$$\dot{x} = f(x), \quad x(0) = x_0$$

defined on its maximal interval of existence $I(x_0)$. Then for $t \in I(x_0)$, the mapping $\phi_t : E \to E$ defined by $\phi_t(x_0) = \phi(t, x_0)$ is called the flow of the differential equation.

Definition 3.4. For any $x_0 \in \mathbb{R}^n$, let $\phi_t(x_0)$ be the flow of the differential equation through x_0. The local stable set S and the local unstable set W of x_0 corresponding to a neighborhood V of x_0 are defined by

$$\begin{aligned} S = S(0) &= \{x_0 \in \mathbb{R}^n : \phi_t(x_0) \in V, t \geq 0, \text{ and } \phi_t(x_0) \to 0 \text{ as } t \to \infty\} \\ W = W(0) &= \{x_0 \in \mathbb{R}^n : \phi_t(x_0) \in V, t \leq 0, \text{ and } \phi_t(x_0) \to 0 \text{ as } t \to -\infty\}. \end{aligned}$$

These local sets are submanifolds of $\mathbb{R}^n$ in a sufficiently small neighborhood V of x_0.

Now we give the theorems of this section.

Theorem 3.2 (The Local Stable-Unstable Manifold Theorem [9, 21])**.** *Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system $\dot{x} = f(x)$. Suppose that $f(0) = 0$ and that $Df(0)$ (the Jacobian matrix) has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold S (stable manifold) tangent to the stable subspace E^S of the linear system $\dot{x} = Ax$ at x_0 such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $x_0 \in S$,*

$$\lim_{t\to\infty} \phi_t(x_0) = 0;$$

and there exists an $n-k$ dimensional differentiable manifold W (unstable manifold) tangent to the unstable subspace E^W of $\dot{x} = Ax$ at x_0 such that for all $t \leq 0$, $\phi_t(W) \subset W$ and for all $x_0 \in W$,

$$\lim_{t\to-\infty} \phi_t(x_0) = 0.$$

Notice that the manifolds S and W are unique.

Theorem 3.3 (The Center Manifold Theorem [9, 21])**.** *Let E be an open subset of $\mathbb{R}^n$ containing the origin and $r \geq 1$, also let $f \in C^r(E)$, i.e. f is a continuos differentiable function on E of order r. Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part, j eigenvalues with positive real part, and $l = n - k - j$ eigenvalues with zero real part. Then there exists an l-dimensional center manifold $W^c(0)$ of class C^r tangent to the center subspace E^c of $\dot{x} = Ax$ at 0 which is invariant under the flow ϕ_t of $\dot{x} = f(x)$.*

In general the center manifold $W^c(0)$ is not unique.

Theorem 3.4 (The Hartman-Grobman Theorem [9, 21]). *Let E be an open subset of $\mathbb{R}^n$ containing the origin, let ϕ_t be the flow of the nonlinear system $\dot{x} = f(x)$. Suppose that $f(0) = 0$, i.e., the origin is an equilibrium point of the dynamical system; and that the Jacobian matrix evaluated at the origin, $A = Df(0)$, has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set W containing the origin onto an open set V also containing the origin such that for each $x_0 \in W$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $x_0 \in W$ and $t \in I_0$*

$$H \circ \phi_t(x_0) = e^{At} H(x_0)\,;$$

that is, H maps trajectories of the nonlinear system $\dot{x} = f(x)$ near the origin onto trajectories of $\dot{x} = Ax$ near the origin and preserves the parametrization.

The following argument show that for any matrix $A = U^\top T_A U$, there exists an homeomorphism $\widehat{H} = UH$ such that for an open set W containing the origin onto an open set V also containing the origin such that for each $x_0 \in W$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $x_0 \in W$ and $t \in I_0$

$$\widehat{H} \circ \phi_t(x_0) = e^{T_A t} \widehat{H}(x_0)\,;$$

the last equality is consequence of the Hartman-Grobman Theorem and the fact $Ue^{At} = e^{T_A t}U$, i.e., $\widehat{H}$ maps trajectories of the nonlinear system $\dot{x} = f(x)$ near the origin onto trajectories of $\dot{x} = T_A x$ near the origin and preserves the parametrization.

The following definition play a key roll in Physics and in section 5.

Definition 3.5. If G is a group and X is a a set, then a (left) *group action* of G on X is a binary function $G \times X \to X$, denoted by

$$(g, x) \mapsto g \cdot x$$

which satisfies the following two axioms:

1. $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$;
2. $e \cdot x = x$ for every $x \in X$ (where e denotes the identity element of G).

The set X is called a (left) G-set. The group G is said to act on X (on the left).

The *action is faithful* (or effective) if for any two different $g, h \in G$ there exists an $x \in X$ such that $g \cdot x \neq h \cdot x$; or equivalently, if for any $g \neq e$ in G there exists an $x \in X$ such that $g \cdot x \neq x$.

The *action is free* or semiregular if for any two different $g, h \in G$ and all $x \in X$ we have $g \cdot x \neq h \cdot x$; or equivalently, if $g \cdot x = x$ for some x implies $g = e$.

For every $x \in X$, we define the *stabilizer subgroup* of x (also called the isotropy group or little group) as the set of all elements in G that fix x:

$$G_x = \{g \in G : g \cdot x = x\}$$

This is a subgroup of G, though typically not a normal one. The action of G on X is free if and only if all stabilizers are trivial.

4 New mathematical tools

In the present section, we show a simple extensions of the Local Stable-Unstable Manifold Theorem and the Center Manifold Theorem, using the tools presented in section 2 and section 3. These extensions are tools that will be used in section 5, where we will present the results on preservation of synchronization in networks.

The following Proposition is a simple extension of the Local Stable-Unstable Manifold Theorem for the action of the group Λ_U on the matrix A and the vector field $f(x)$, where $A = UT_AU^\top$ with T_A an upper triangular matrix and $U^\top U = U^\top = I$.

Proposition 4.1. *Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system $\dot{x} = f(x) = Ax + g(x)$. Suppose that $f(0) = 0$ and that $A = Df(0)$ has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part, i.e., the origin is an hyperbolic fixed point. Then for each matrix $M \in \Lambda_U$ there exists a k-dimensional differentiable manifold S_M tangent to the stable subspace E_M^S of the linear system $\dot{x} = MAx$ at 0 such that for all $t \geq 0$, $\phi_{M,t}(S_M) \subset S_M$ and for all $x_0 \in S_M$,*

$$\lim_{t\to\infty} \phi_{M,t}(x_0) = 0,$$

where $\phi_{M,t}$ be the flow of the nonlinear system $\dot{x} = MAx + g(x)$; and there exists an $n - k$ dimensional differentiable manifold W_M tangent to the unstable subspace E_M^W of $\dot{x} = MAx$ at 0 such that for all $t \leq 0$, $\phi_{M,t}(W_M) \subset W_M$ and for all $x_0 \in W_M$,

$$\lim_{t\to-\infty} \phi_{M,t}(x_0) = 0.$$

An interesting property is that Proposition 4.1 is valid for each $\overline{g} \in C^1(E)$ such that $\dot{x} = \overline{f}(x) = Ax + \overline{g}(x)$ and

$$\frac{\|\overline{g}(x)\|_2}{\|x\|_2} \to 0 \text{ as } \|x\|_2 \to 0.$$

In consequence, the set of matrices Λ_U generates the action of the group Λ_U on the set of the hyperbolic nonlinear systems (formally on the set of the hyperbolic vector fields $f \in C^1(E)$) $\dot{x} = \overline{f}(x) = Ax + \overline{g}(x)$ with $\overline{g} \in C^1(E)$ and

$$A \in \Omega_U \equiv \left\{ P \in \mathbb{R}^{n\times n} : P = U^\top T_P U \text{ with } T_P \text{ any upper triangular matrix} \right\}$$

satisfying the last condition, where U is a fixed unitary matrix, this action is faithful and free. The former action is generated by the action of the group Λ_U on the set Ω_U. The first action preserves hyperbolic nonlinear systems and dimension of the stable and unstable manifolds, i.e, an hyperbolic nonlinear systems ($\dot{x} = Ax + \overline{g}(x)$) is mapped in a hyperbolic nonlinear systems ($\dot{x} = MAx + \overline{g}(x)$), and $\dim S = \dim S_M$ and $\dim W = \dim W_M$.

Proof. Consider a matrix A with the decomposition $A = U^\top T_A U$ where T_A is an upper triangular matrix and U is an unitary n by n matrix, by Lemma 2.1 this decomposition always exists, and the decomposition $M = U^{\top} T_M U \in \Lambda_U$. Then $MA = U^{\top} T_M T_A U$ and the eigenvalues of the matrix MA are the product of the eigenvalues of the matrices M and A, respectively. Since each matrix $M \in \Lambda_U$ has all diagonal elements strictly positive, then the matrix MA has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Therefore, the result is a consequence of the Stable-Unstable Manifold Theorem. $\square$

Notice that given an particular nonlinear system, the stable and unstable manifolds S and W are unique, then for each matrix $M \in \Lambda_U$ there exist an unique pair of manifolds (S_M, W_M) in such way that it is possible to define a pair of functions in the following form

$$\begin{aligned}&\Theta : \Lambda_U \times Man_S \to Man_S\\&\Theta(M, S) = S_M\end{aligned}$$

$$\begin{aligned}&\Phi : \Lambda_U \times Man_W \to Man_W\\&\Phi(M, W) = W_M\end{aligned}$$

where Man_S is the set of stable manifolds and Man_W is the set of unstable manifolds for autonomous nonlinear systems.

Notice that if $A = Df(0)$ is an *stable matrix*, i.e., A has all the n eigenvalues with negative real part, then the origin of the nonlinear system $\dot{x} = MAx + \widehat{g}(x)$ is asymptotically stable; and if $A = Df(0)$ is an *unstable matrix*, i.e., A has $n-k$ (with $n > k$) eigenvalues with positive real part, then the origin of the nonlinear system $\dot{x} = MAx + \widehat{g}(x)$ is unstable.

As an extension of the local Stable-Unstable Manifold Theorem in terms of the Kronecker product of matrices in Λ_N and the matrix A of the vector field $f(x)$ we present the following Proposition.

Proposition 4.2.

1. *Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system $\dot{x} = f(x) = Ax + g(x)$. Suppose that $f(0) = 0$ and that $A = Df(0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part, i.e., the origin is a hyperbolic fixed point. Now take a fixed continuously differentiable function*

$$F : C^1(E) \to C^1(\overline{E})$$

such that $F(g) = \widehat{g}$ where $\widehat{g} : \overline{E} \subset \mathbb{R}^{mn} \to \mathbb{R}^{mn}$ is a fixed continuously differentiable function with domain all $C^1(E)$; moreover $\widehat{g} \in C^1(\overline{E})$ with $\overline{E}$ an open subset of $\mathbb{R}^n$ containing the origin such that

$$\frac{\|\widehat{g}(x)\|_2}{\|x\|_2} \to 0 \text{ as } \|x\|_2 \to 0.$$

Then for each matrix $M \in \Lambda_N$ of $m \times m$, there exists a mk-dimensional differentiable manifold $S_{M\otimes A}$ tangent to the stable subspace $E^S_{M\otimes A}$ of the linear system

$\dot{x} = (M \otimes A)\, x$ at 0 such that for all $t \geq 0$, $\phi_{M\otimes A,t}(S_{M\otimes A}) \subset S_{M\otimes A}$ and for all $x_0 \in S_{M\otimes A}$,

$$\lim_{t\to\infty} \phi_{M\otimes A,t}(x_0) = 0,$$

where $\phi_{M\otimes A,t}$ be the flow of the nonlinear system $\dot{x} = (M \otimes A)\, x + \widehat{g}(x)$; and there exists an $m\,(n-k)$ dimensional differentiable manifold $W_{M\otimes A}$ tangent to the unstable subspace $E^W_{M\otimes A}$ of $\dot{x} = (M \otimes A)\, x$ at 0 such that for all $t \leq 0$, $\phi_{M\otimes A,t}(W_{M\otimes A}) \subset W_{M\otimes A}$ and for all $x_0 \in W_{M\otimes A}$,

$$\lim_{t\to -\infty} \phi_{M\otimes A,t}(x_0) = 0;$$

2. *Also, there exists an function of the group Λ_N and the set of all the autonomous hyperbolic nonlinear systems of dimension n (hyperbolic vector fields of dimension n) denoted by Γ_n, to the set Γ_{mn} of all the autonomous hyperbolic nonlinear systems of dimension mn (hyperbolic vector fields of dimension mn), this function (which is similar to an action of the group Λ_N on the set Γ_n) is defined as follows*

$$\vartheta : \Lambda_N \times \Gamma_n \to \Gamma_{mn}$$
$$\vartheta(M, Ax + g(x)) = (M \otimes A)\, x + \widehat{g}(x)$$

and the new nonlinear system is

$$\begin{aligned} \dot{x} &= \vartheta(M, Ax + g(x)) \\ \dot{x} &= (M \otimes A)\, x + \widehat{g}(x). \end{aligned}$$

which satisfies the following two axioms:

(a) $(gh) \cdot z = g \bullet (h \cdot z)$ *for all $g, h \in \Lambda_N$ and $z \in \Gamma_n$;*

(b) For every $z \in \Gamma_n$ there exists an unique $\widehat{z} \in \Gamma_{mn}$ such that $e \cdot z = \widehat{z}$ and $h \bullet \widehat{z} = h \cdot z$ (e denotes the identity element of Λ_N, i.e., e is the identity matrix I_m of $m \times m$).

Where z is associated with $Ax + g(x)$ (denoted by $z \triangleq Ax + g(x)$); $h \cdot z$ means $(M_h \otimes A)\, x + \widehat{g}(x)$ (denoted by $h \cdot z \triangleq (M_h \otimes A)\, x + \widehat{g}(x)$); gh is associated with the usual product of matrices $M_g M_h$, i.e., $gh \triangleq M_g M_h$ and $e \cdot z$ means $(I_m \otimes A)\, x + \widehat{g}(x)$, i.e., ($e \cdot z \triangleq (I_m \otimes A)\, x + \widehat{g}(x)$), and $g \bullet (h \cdot z)$ means$(M_g \otimes I_n)\,(M_h \otimes A)\, x + \widehat{g}(x)$ (denoted by $g \bullet (h \cdot z) \triangleq (M_g \otimes I_n)\,(M_h \otimes A)\, x + \widehat{g}(x)$).

Proof.

1. Consider a matrix A with eigenvalues λ_i for $i = 1, 2, \ldots, n$, and the matrix M with eigenvalues μ_j for $j = 1, 2, \ldots, m$. Then the eigenvalues of the matrix $M \otimes A$ are the eigenvalues are the mn numbers $\lambda_i \mu_j$, and taking account that $\mu_j > 0$ for each $j = 1, 2, \ldots, m$. Therefore, the matrix $M \otimes A$ has mk eigenvalues with negative real part and $m\,(n-k)$ eigenvalues with positive real part. Now, the result is a consequence of the Stable-Unstable Manifold Theorem.

2. The function $\vartheta : \Lambda_N \times \Gamma_n \to \Gamma_{mn}$ is well defined, since $F : C^1(E) \to C^1(\overline{E})$ is a fixed function, then given $g(x)$, the vector field $\widehat{g}(x)$ is unique, and for a fixed matrix $M_h \in \Lambda_N$, then $M_h \otimes_{-} : R^{n\times n} \to \mathbb{R}^{mn\times mn}$ is a fixed function and the matrix $M_h \otimes A$ is unique.

   Axiom (a): Since Λ_N is a multiplicative group if M_g, $M_h \in \Lambda_N$, then $M_g M_h \in \Lambda_N$. Now, by Lemma 2.2, we have that for all g, $h \in \Lambda_N$ and $z \in \Gamma_n$

$$
\begin{aligned}
(gh) \cdot z \mathrel{\hat{=}} (M_g M_h \otimes A)\, x + \widehat{g}(x) = \\
(M_g \otimes I_n)\,(M_h \otimes A)\, x + \widehat{g}(x) \mathrel{\hat{=}} g \bullet (h \cdot z).
\end{aligned}
$$

   Axiom (b): For every $z \in \Gamma_n$ there exists an unique $\widehat{z} \in \Gamma_{mn}$ such that $e \cdot z \mathrel{\hat{=}} (I_m \otimes A)\, x + \widehat{g}(x) = \widehat{z}$, then by the Lemma 2.2

$$
\begin{aligned}
h \bullet \widehat{z} \mathrel{\hat{=}} (M_h \otimes I_n)\,(I_m \otimes A)\, x + \widehat{g}(x) = \\
(M_h \otimes A)\, x + \widehat{g}(x) \mathrel{\hat{=}} h \cdot z.
\end{aligned}
$$

$\square$

Notice that if $A = Df(0)$ is *stable matrix*, i.e., A has all the n eigenvalues with negative real part, then the origin of the nonlinear system $\dot{x} = (M \otimes A)\, x + \widehat{g}(x)$ is asymptotically stable; and if $A = Df(0)$ is *unstable matrix*, i.e., A has $n - k$ $(n > k)$ eigenvalues with positive real part, then the origin of the nonlinear system $\dot{x} = (M \otimes A)\, x + \widehat{g}(x)$ is unstable.

The following Proposition is an extension of the Center Manifold Theorem, similar to Proposition 4.1 and Proposition 4.2.

Proposition 4.3. *Let $f \in C^r(E)$ where E is an open subset of $\mathbb{R}^n$ containing the origin and $r \geq 1$. Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part, j eigenvalues with positive real part, and $l = n - k - j$ eigenvalues with zero real part. Then*

1. *For each matrix $M \in \Lambda_U$ there exists a m-dimensional differentiable center manifold $W^c_M(0)$ of class C^r tangent to the center subspace E^c_M of the linear system $\dot{x} = MAx$ at 0 which is invariant under the flow $\phi_{M,t}$ of the nonlinear system $\dot{x} = MAx + g(x)$.*

2. *If taken a fixed continuously differentiable function*

$$
\widehat{F} : C^r(E) \to C^r(\overline{E})
$$

   such that $F(g) = \widehat{g}$ where $\widehat{g} : \overline{E} \subset \mathbb{R}^{mn} \to \mathbb{R}^{mn}$ is a fixed continuously differentiable function with domain all $C^r(E)$; moreover $\widehat{g} \in C^r(\overline{E})$ with $\overline{E}$ an open subset of $\mathbb{R}^n$ containing the origin such that

$$
\frac{\|\widehat{g}(x)\|_2}{\|x\|_2} \to 0 \text{ as } \|x\|_2 \to 0.
$$

Then for each matrix $M \in \Lambda_N$ of $m \times m$, there exists a ml-dimensional differentiable center manifold $W^c_{M\otimes A}(0)$ tangent to the center subspace $E^S_{M\otimes A}$ of the linear system $\dot{x} = (M \otimes A)\, x$ at 0 which is invariant under the flow $\phi_{M\otimes A,t}$ of the nonlinear system $\dot{x} = (M \otimes A)\, x + \widehat{g}(x)$.

Proof. The proof is similar to proof of Proposition 4.1 and Proposition 4.2, and we make use of the Center Manifold Theorem. □

Also, there exists a similar function $\widehat{\vartheta}$ to ϑ which satisfies the axiom (a) and axiom (b) of Proposition 4.2. However, in this case there does not exist similar functions to Θ and Φ, due to that in general a center manifold is not unique.

Notice that in this case, if the matrix A has $l = n-k-j \neq 0$ eigenvalues with zero real part, then the origin of the nonlinear systems $\dot{x} = MAx+\widehat{g}(x)$ and $\dot{x} = (M \otimes A)\, x+\widehat{g}(x)$ are not asymptotically stable.

Propositions 4.1, 4.2, 4.3 generalize Proposition 3 in [13], and give new tools for preservation of basic properties of dynamical systems, some of these properties are the stability and instability.

5 Preservation of synchronization in networks

Now, we will present that it is possible to preserve synchronization even thought the dimension of the system changes by the action of a class of transformation on the linear part of a chaotic nonlinear dynamical system.

Consider the following two n-dimensional chaotic systems:

$$\begin{aligned} \dot{x} &= Ax + g(x) \\ \dot{y} &= Ay + f(y) + u(t) \end{aligned}$$

where $A \in R^{n\times n}$ is a constant matrix, $f,\ g : R^n \to R^n$ are continuous nonlinear functions and $u \in R^n$ is the control input. The problem of synchronization considered in this section is the complete-state exact synchronization, that is, the master system and the slave system are synchronized by designing an appropriate nonlinear state feedback control $u(t)$ which is attached to the slave system such that:

$$\lim_{t\to\infty} \|x(t) - y(t)\| \to 0$$

where $\|\cdot\|$ is the Euclidean norm of a vector.

Considering the error state vector $e = y - x \in R^n$, $f(y) - g(x) = L(x, y)$ and an error dynamics equation:

$$\dot{e} = Ae + L(x, y) + u(t).$$

Based in the active control approach [5], to eliminate the nonlinear part of the error dynamics, and choosing $u(t) = Bv(t) - L(x, y)$, where B is a constant gain vector which is selected such that (A, B) be controllable, we obtain:

$$\dot{e} = Ae + Bv(t).$$

Notice that the original synchronization problem is equivalent to the problem of stabilizing the zero-input solution of the last system by a suitable choice of the state feedback control.

Since the pair (A, B) is controllable one such suitable choice for state feedback is a linear-quadratic state-feedback regulator [2], which minimizes the quadratic cost function

$$J(u(t)) = \int_0^\infty \left(e(t)^\top Q e(t) + v(t)^\top R v(t)\right) dt$$

where Q and R are positive semi-definite and a positive definite weighting matrices, respectively. The state-feedback law is given by $v = -Ke$ with $K = R^{-1}B^\top S$ and S the solution to the Riccati equation

$$A^\top S + SA - SBR^{-1}B^\top + Q = 0.$$

This state-feedback law renders the error equation to $\dot{e} = (A - BK)e$, with $(A - BK)$ a Hurwitz matrix[1]. The linear quadratic regulator (LQR) is a well-known design technique that provides practical feedback gains [2]. An interesting property of (LQR) is robustness.

Now consider $T \in R^{m\times m}$ be a matrix with strictly positive eigenvalues, and suppose that the following two nm-dimensional systems are chaotic:

$$\begin{aligned}\dot{x} &= (T \otimes A)\, x + \widehat{g}(x)\\ \dot{y} &= (T \otimes A)\, y + \widehat{f}(y) + \widehat{u}(t)\end{aligned}$$

for some $\widehat{f}, \widehat{g} : R^{nm} \to R^{nm}$ continuous nonlinear functions and $\widehat{u} \in R^{nm}$ is the control input. Then based on Proposition 4.2, and the former procedure, we have that $\widehat{u}(t) = \widehat{\theta}(t) - \widehat{L}(x, y)$ stabilizes the zero solution of the error dynamics system, where $\widehat{\theta}(t) = -\left(BK \otimes T\right) e$, i.e., the resultant system

$$\begin{aligned}\dot{e} &= (T \otimes A)\, e + \widehat{\theta}(t)\\ \dot{e} &= (T \otimes A - T \otimes BK)\, e\end{aligned}$$

is asymptotically stable. Notice that using Lemma 2.2 and $K = -R^{-1}B^\top S$, we obtain that:

$$\begin{aligned}\dot{e} &= (T \otimes (A + BK))\, e\\ \dot{e} &= \left(T \otimes \left(A - BR^{-1}B^\top S\right)\right) e\end{aligned}$$

The original control $u(t) = BKe - L(x, y)$ is preserved in its linear part by the Kroneker product $T\otimes(\cdot)$ and the new control is given by $\widehat{u}(t) = -\left(T \otimes BK\right) e - \widehat{L}(x, y)$. Therefore, we can interpreted the last procedure as one in which the controller $u(t)$ that achieves the synchronization in the two original systems is preserved by the transformation $T\otimes(\cdot)$ so that

[1]A Hurwitz matrix is a matrix for which all its eigenvalues are such that their real part is strictly less than zero

$\widehat{u}(t)$ achives the synchronization in the two resultant systems after of the transformation. A similar procedure is possible if we consider the transformation $(\cdot) \otimes T$.

In general, under the transformations $(A, g) \to (MA, \overline{g})$ or $(A, g) \to (M \otimes A, \overline{g})$, and under the hypothesis of the existence of a constant state feedback $U = -Kx$ which achieves synchronization of the original chaotic systems, and also that the transformed systems are chaotic, then synchronization can be preserved.

The main contribution in this section does not deal with a better synchronization methodology, rather it deals with the fact that synchronization is preserved when the underlying chaotic dynamical system changes from a lower dimension to a higher dimension. This issue is important for the case of networking systems. In this section, the transformed system can be interpreted as a network in which there has been an increase in the number of nodes.

6 Synchronization of transformed Chua's circuit

In this section using a known chaotic system as a benchmark for the result shown in Section 5 some simulations are presented.

6.1 Synchronization of modified Chua's circuit

The known chaotic system which we will use to show the posibility to preserve synchronization is the modified Chua's circuit described in [16]:

$$
\begin{aligned}
\dot{x}_1 &= p\left(x_2 - \frac{(2x_1^3 - x_1)}{7}\right) \\
\dot{x}_2 &= x_1 - x_2 + x_3 \\
\dot{x}_3 &= -qx_2
\end{aligned}
$$

which has a chaotic attractor. In order to observe synchronization behavior in [16], they have two modified Chua's circuits arrange as a Master/Slave configuration. The Master and the Slave systems are almost identical, the only difference is that the Slave systems includes an extra term which is used for the purpose of synchronization with the Master system. The initial condition for the two systems are different. The two modified Chua's circuits are described, respectively, by the following equations.

The master system is given by

$$
\begin{aligned}
&\dot{x}_1 = p\left(x_2 - \frac{2x_1^3 - x_1}{7}\right) \\
&\dot{x}_2 = x_1 - x_2 + x_3 \\
&\dot{x}_3 = -qx_2
\end{aligned}
$$

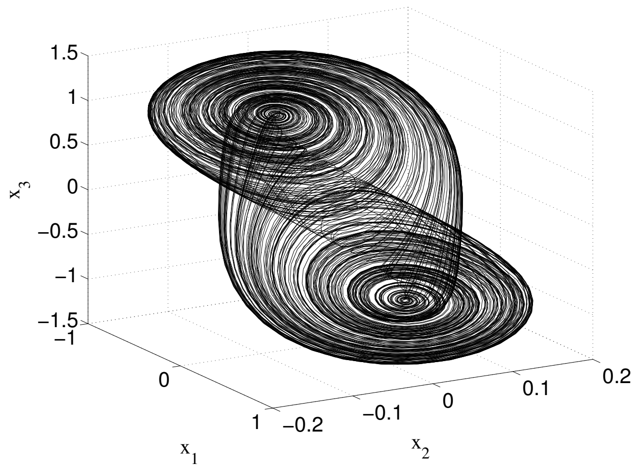

Figure 1: Modified Chua's circuit (Master), Initial conditions $x = \left[\begin{array}{ccc} 0.02 & 0.05 & 0.04 \end{array}\right]$.

and the slave system is a copy of the master system with a control function $u(t)$ to be determined in order to synchronize the two systems.

$$\begin{aligned}
\dot{y}_1 &= p\left(y_2 - \frac{2y_1^3 - y_1}{7}\right) + u_1 \\
\dot{y}_2 &= y_1 - y_2 + y_3 + u_2 \\
\dot{y}_3 &= -qy_2 + u_3
\end{aligned}$$

Considering the errors $e_1 = x_1 - y_1$, $e_2 = x_2 - y_2$, $e_3 = x_3 - y_3$, then the error dynamics can be written as:

$$\begin{aligned}
\dot{e}_1 &= \frac{p}{7}e_1 + pe_2 - \frac{2p}{7}\left(y_1^3 - x_1^3\right) + u_1(t) \\
\dot{e}_2 &= e_1 - e_2 + e_3 + u_2(t) \\
\dot{e}_z &= -qe_2 + u_3(t)
\end{aligned}$$

Introducing the matrices:

$$A = \begin{pmatrix} \frac{p}{7} & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{pmatrix},\ L(x,y) = \begin{pmatrix} -\frac{2p\left(y_1^3 - x_3^2\right)}{7} \\ 0 \\ 0 \end{pmatrix},\ u = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}$$

and the vector B is selected such that (A, B) is controllable: $B = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^T$. Now the LQR controller is obtained by using weighting matrices $Q = I$ and $R = B^\top B$ and parameter values $p = 10$ and $q = \frac{100}{7}$. The state feedback matrix is given by $K = \begin{pmatrix} 3.8676 & 5.1415 & 1.2932 \end{pmatrix}$.

In Figure 1 and Figure 2 the trajectories for the solution of the master system and slave system are shown. In Figure 3 the absolute value for the error between the master and slave

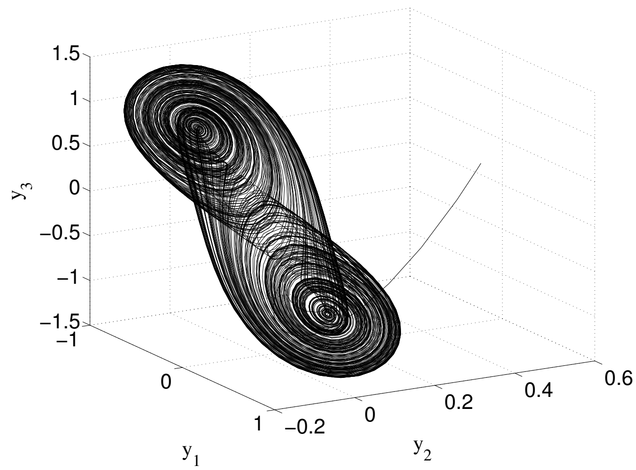

Figure 2: Modified Chua's circuit (Slave), Initial conditions $x = \left[\begin{array}{ccc} 0.2 & 0.5 & 0.4 \end{array}\right]$.

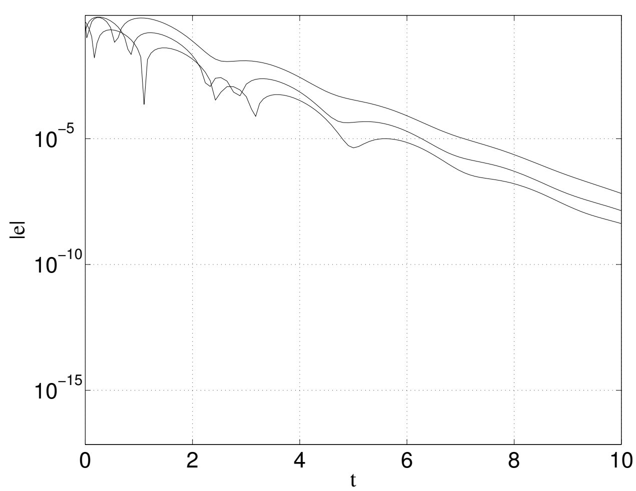

Figure 3: Magnitude of the error $|e| = |y - x|$ between the Master and the slave systems.

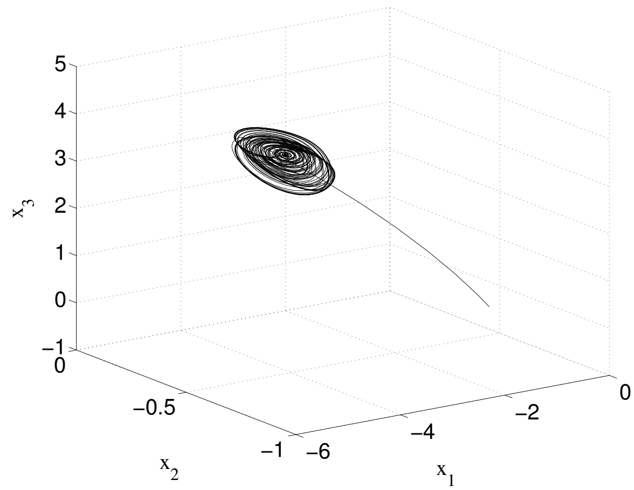

Figure 4: Transformation of the first kind for the modified Chua's model (Master). States x_1, x_2 and x_3 with initial conditions -0.2, -0.5, -0.4, respectively.

systems are shown in a semi-logarithmic plot to emphasize the fact that the error converges to zero and therefore the synchronization between the Master and Slave systems.

6.2 Transformation of the first kind

For the transformed systems of the first kind we have the master and slave systems as

$$\begin{aligned}\dot{x} &= (T \otimes A)x + \begin{bmatrix} -\frac{2x_1^3}{7} & -\frac{2x_2^3}{7} & -\frac{2x_3^3}{7} & -\frac{2x_4^3}{7} & 0 & 0 \end{bmatrix}^\top, \\ \dot{y} &= (T \otimes A)y + \begin{bmatrix} -\frac{2y_1^3}{7} & -\frac{2y_2^3}{7} & -\frac{2y_3^3}{7} & -\frac{2y_4^3}{7} & 0 & 0 \end{bmatrix}^\top + u\end{aligned}$$

Considering the error vector $e = y - x$, then the error dynamics can be written as:

$$\dot{e} = (T \otimes A)e + L(x, y) + u$$

with $u = -M(x, y) + v$ and $v = -(BK \otimes T)e$ and

$$\begin{aligned}A &= \begin{bmatrix} \frac{p}{7} & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \\ L(x, y) &= \begin{bmatrix} l_1 & l_2 & l_3 & l_4 & 0 & 0 \end{bmatrix}^\top, \quad l_i = -\frac{2\left(y_i^3 - x_i^3\right)}{7} \\ B &= \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^\top\end{aligned}$$

We used K the same as in subsection 6.1.

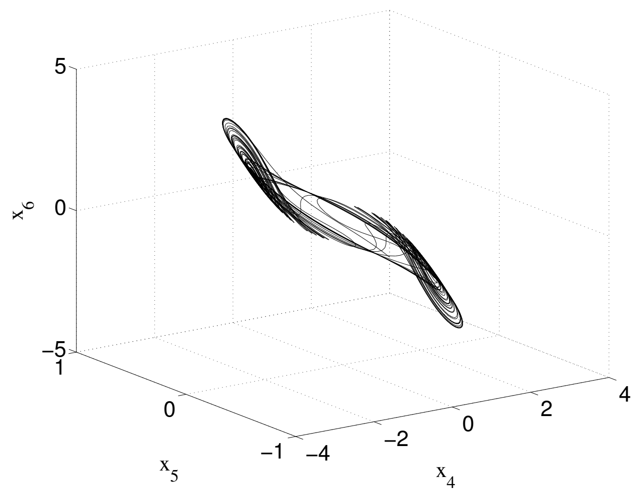

Figure 5: Transformation of the first kind for the modified Chua's model (Master). States x_4, x_5 and x_6 with initial conditions -0.2, -0.5, -0.4, respectively.

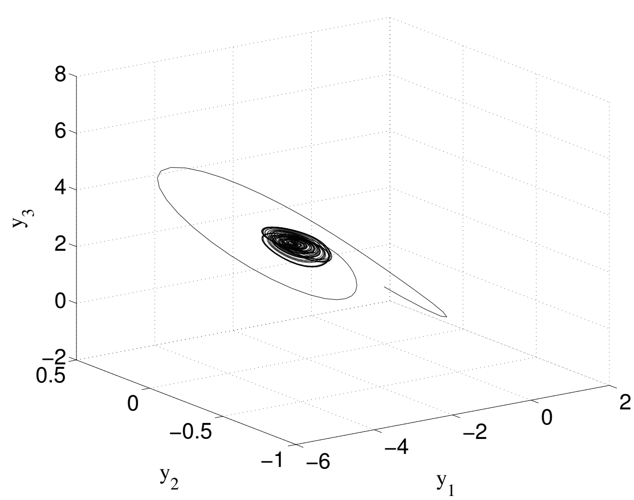

Figure 6: Transformation of the first kind for the modified Chua's model (Slave). States y_1, y_2 and y_3 with initial conditions 0.002, 0.005, 0.004, respectively.

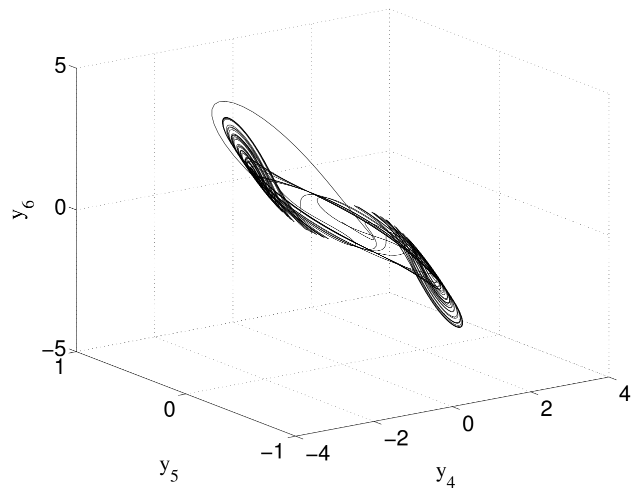

Figure 7: Transformation of the first kind for the modified Chua's model (Slave). States y_4, y_5 and y_6 with initial conditions 0.002, 0.005, 0.004, respectively.

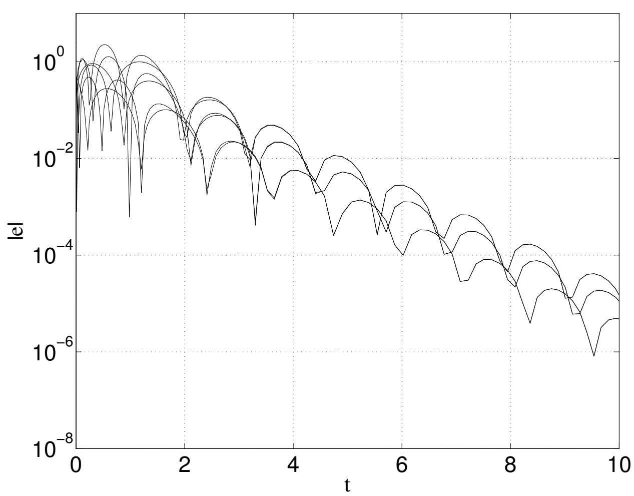

Figure 8: Magnitude of the error $|e| = |y - x|$ between the Master and the slave systems.

In Figure 4 and Figure 5 the trajectories for the master system are shown; in Figure 6 and Figure 7 the trajectories for the slave system are presented. In Figure 8 the absolute value for the error between the master and slave systems are shown in a semi-logarithmic plot to emphasize the fact that the error converges to zero.

Notice that the transformed systems are of higher dimension than the original modified Chua's circuit model, and also the transform systems are chaotic. It is important to mention that the nonlinear part was chosen such that the transformed systems are chaotic, since the transformation is not applied to the higher order terms of the vector field.

The same transformation that preserves stability also preserves the controller and therefore synchronization for the transformed modified Chua's circuit.

6.3 Transformation second kind

For the transformed systems of the second kind we have the master and slave systems as

$$\begin{aligned}\dot{x} &= (M_1 M_2 \otimes A)x + \left[\begin{array}{cccccc} -\frac{2x_1^3}{7} & -\frac{2x_2^3}{7} & -\frac{2x_3^3}{7} & -\frac{2x_4^3}{7} & -\frac{2x_5^3}{7} & -\frac{2x_6^3}{7}\end{array}\right]^\top, \\ \dot{y} &= (M_1 M_2 \otimes A)y + \left[\begin{array}{cccccc} -\frac{2y_1^3}{7} & -\frac{2y_2^3}{7} & -\frac{2y_3^3}{7} & -\frac{2y_4^3}{7} & -\frac{2y_5^3}{7} & -\frac{2y_6^3}{7}\end{array}\right]^\top + u\end{aligned}$$

Considering the error vector $e = y - x$, then the error dynamics can be written as:

$$\dot{e} = (M_1 M_2 \otimes A)e + L(x, y) + u$$

with $u = -M(x, y) + v$ and $v = -(M_1 M_2 \otimes BK)e$ and

$$\begin{aligned} A &= \left[\begin{array}{ccc} \frac{p}{7} & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{array}\right], \quad M_1 = \left[\begin{array}{cc} 3.1532 & -3.4839 \\ 1.5161 & -1.4532 \end{array}\right], \\ M_2 &= \left[\begin{array}{cc} 2.2764 & 1.1708 \\ 0.17082 & 2.7236 \end{array}\right], \\ L(x, y) &= \left[\begin{array}{cccccc} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \end{array}\right]^\top, \quad l_i = -\frac{2\left(y_i^3 - x_i^3\right)}{7} \\ B &= \left[\begin{array}{ccc} 0 & 1 & 1 \end{array}\right]^\top \end{aligned}$$

notice that M_1 and M_2 are equivalent to upper triangular matrices

$$T_1 = \left[\begin{array}{cc} 1 & -0.5 \\ 0 & 0.7 \end{array}\right], \quad T_2 = \left[\begin{array}{cc} 3 & 1 \\ 0 & 2 \end{array}\right],$$

by the similarity transformation

$$U = \left[\begin{array}{cc} 0.85065 & -0.52573 \\ 0.52573 & 0.85065 \end{array}\right].$$

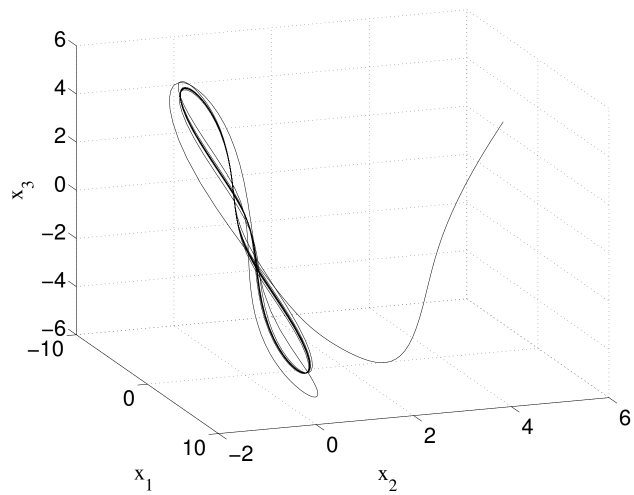

Figure 9: Transformation of the second kind for the modified Chua's model (Master). States x_1, x_2 and x_3 with initial conditions 2, 5, 4, respectively.

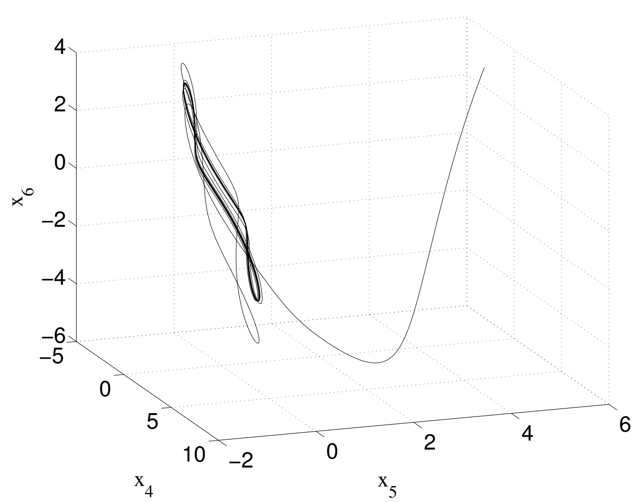

Figure 10: Transformation of the second kind for the modified Chua's model (Master). States x_4, x_5 and x_6 with initial conditions 2, 5, 4, respectively.

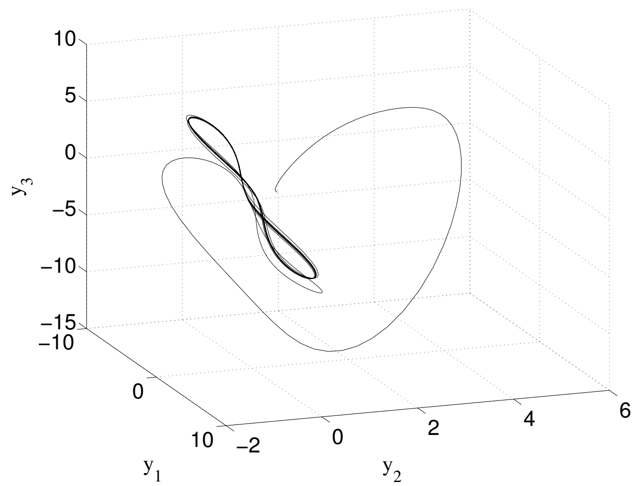

Figure 11: Transformation of the second kind for the modified Chua's model (Slave). States y_1, y_2 and y_3 with initial conditions 0.2, 0.5, 0.4, respectively.

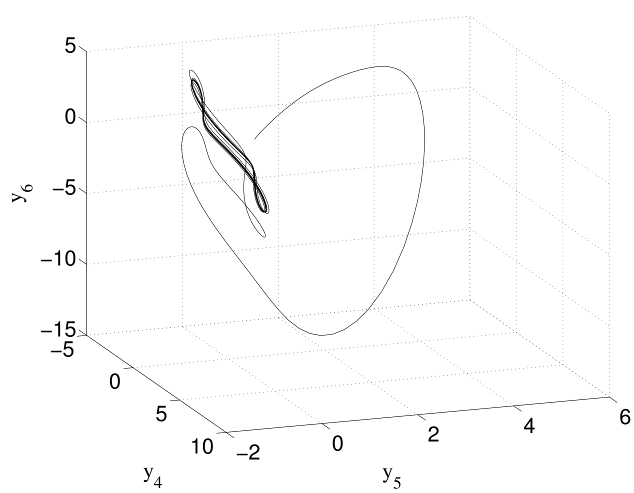

Figure 12: Transformation of the second kind for the modified Chua's model (Slave). States y_4, y_5 and y_6 with initial conditions 0.2, 0.5, 0.4, respectively.

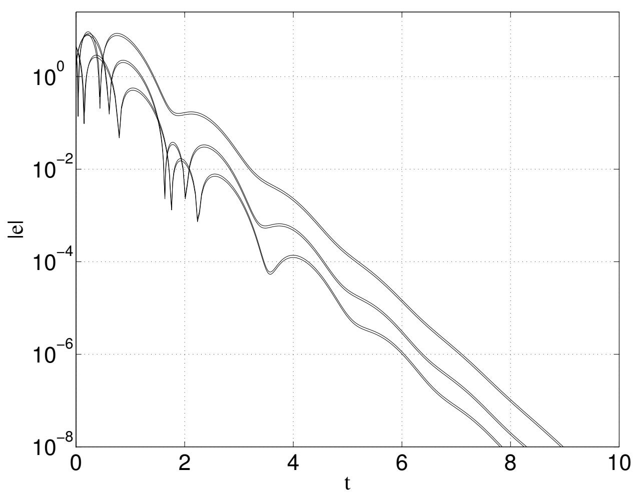

Figure 13: Magnitude of the error $|e| = |y - x|$ between the Master and the slave systems.

We used K the same as in subsection 6.1.

In Figure 9 and Figure 10 the trajectories for the master system are shown; in Figure 11 and Figure 12 the trajectories for the slave system are presented. In Figure 13 the absolute value for the error between the master and slave systems are shown in a semi-logarithmic plot to emphasize the fact that the error converges to zero.

The resulting trajectories seem to preserved a region of attraction, nevertheless at this point it is not clear if chaos is preserved or not. As a future work it would be important to characterize which kind of mechanism preserve the chaotic behavior of a system even in the event of a a change on the dimension of the system.

7 Conclusion

The preservation of the stable behavior in chaotic synchronization is studied from an extension of the local stable-unstable manifold theorem and an extension of the center manifold theorem based in the preservation of the signature of the linear part of the vector fields in nonlinear dynamical systems. Furthermore, under the hypothesis that given a chaotic system, its transformed version is also a chaotic system, it is shown that a scheme consisting of a master/slave pair for which a constant state feedback where the controller gain for the slave system is obtained using a linear-quadratic regulator for which the chaotic synchronization is achieved preserving the synchronization even after the master/slave/controller is transformed, notice that the underlying chaotic dynamical system changes from a lower dimension. It is an attempt to study how a given collective dynamics can be preserved when important changes occur in the dynamical system. This issue is important for the case of networking systems. We know that most of real world networks are not stationary, in the sense that they are growing, with new nodes continuously being added to the graph.

This is related to the increase in dimension of the involved network. Therefore it comes out a natural question on how these networks can preserve a given collective dynamics or functioning, while the process of their growth is taking place in time. The transformed system can be interpreted as a network in which there has been an increase in the number of nodes. As an example a modified Chua's circuits is used to show the effectiveness of the proposed method. The results can be extended to other technique for the feedback design, e.g., adaptive, sliding mode, etc.

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