A simple model for Jeans instability in the presence of a constant magnetic field

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Abstract

Jeans instability condition is derived for the case of a low density self-gravitating fully ionized gas in the presence of a constant magnetic field. A new Jeans wave number, substantially enhanced by the magnetic field, is established under a suitable symmetry assumption. Nevertheless, due to our present knowledge regarding the existence of primeval magnetic fields in the Universe, it appears that they are indeed irrelevant for our understanding of large scale structure formation. An estimation for Jeans mass is presented for cosmological magnetic fields.

1 Introduction

Magnetic fields have a significant effect on virtually all astrophysical objects. They are observed in all scales. Close to home, the Earth has a bipolar magnetic field with a strength of $0.3G$ at the equator and $0.6G$ at the poles [2]. Within the interstellar medium, magnetic fields are thought to regulate star formation via the ambipolar diffusion mechanism [9]. Our Galaxy has a typical interstellar magnetic field strength of $\sim 2\mu G$ in both regular ordered and random components. Other spiral galaxies have been estimated to have magnetic field strengths of 5 to $10\mu G$, with fields strengths up to $50\mu G$ found in starburst galaxy nuclei [1]. Also magnetic fields are fundamental to the observed properties of jets and lobes in radio galaxies, and they may be primary elements in the generation of relativistic outflows from accreting massive black holes [2].

Magnetic fields with typical strength of order $1\mu G$ have been measured in the intercluster medium using a variety of techniques. Large variations in the field strength and topology are expected from cluster to cluster, especially when comparing dynamically relaxed clusters to those that have recently undergone a merger. Magnetic fields with strengths of $10 - 40\mu G$ have been observed in some locations [2]. In all cases, the magnetic fields play important role in the energy transport in the intercluster medium and in gas collapse.

On the other hand at the cosmological level the presence or existence of magnetic fields is more controversial. In a recent review on the subject [11] it is firmly asserted that a true cosmological magnetic field is one that cannot be associated with collapsing or virialized structures. Thus the particular role that they may play in the epoch of galaxy formation is rather obscure. Although limits have been placed on the strength of cosmological magnetics fields from Faraday rotation studies, of high redshift sources, anisotropy measurements of the CMB and the light element abundances from nucleosynthesis, the question remains: Is there the possibility that the Jeans mass arising from gravitational instabilities responsible for galaxies formation be modified by the presence of a magnetic field?.

In spite of the dubious background provided by our present knowledge, this question has been tackled since over fifty years ago. In fact, already Chandrasekhar & Fermi [3] reached the conclusion that Jeans criteria for the onset of a hydrodynamic instability is unaffected by a magnetic field in an extended homogeneous gas of infinite conductivity in the presence of an uniform magnetic field. However, in their calculation they assumed that the medium was magnetized and that, within the gas, there existed a fluctuating magnetic field. This problem has been retaken by several other authors in different contexts. In particular Lou [6] studied the problem of gravitational collapse in a magnetized dynamic plasma in the presence of a finite amplitude circular polarized Alfvén wave. This author does find a case in which Jeans wave number k_J is indeed modified by the magnetic field by a term proportional to $[c_0^2 + c_A^2]^{1/2}$ where c_0 is the velocity of sound and $c_A = Bz_0(4\pi\rho_0)^{-1/2}$ is Alfvén wave speed Bz_0 being the z-component of the uniform magnetic field. Other attempts to show that magnetic fields do play an essential of roll in galaxy formation have been performed, Kim, Olinto & Rosner [5], although not specifically addressing the question of a magnetic instability. Tsagas and Maartens [6] have performed a magnetohydrodynamical analysis within a relativistic framework addressing Jeans instabilty on the basis of previous work, Tsagas and Barrow [7] [8].

In view of all these efforts we still feel that the simple question of whether or not a dilute non-magnetized plasma cloud placed in the presence of an external, uniform magnetic field in which density fluctuations are also present due to a fluctuating gravitational field, exhibits a Jeans wave number which is modified by the presence of the field, has not yet been fully discussed in the literature. This is the purpose of the present work. The basic and rather simple formalism is given in § 2. § 3 is devoted to the derivation of the dispersion relation leading to the modified form of k_J and some attempts to place the relevance of the results within a realistic frame for existing magnetic field intensities. Some concluding remarks are given in § 4.

2 Basic Formalism

We start by assuming that the dynamics of the dilute plasma is governed by Euler's equations of hydrodynamics, namely the balance equations for the fluid's mass density $\rho(\vec{r},t)$ and its velocity $u(\vec{r},t)$. Thus,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \tag{1}$$

$$\frac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u}\vec{u}) + \nabla p = \vec{f}_g + \vec{f}_M \tag{2}$$

In equation (2), $\vec{f}_g$ is the force arising from the gravitational field and $\vec{f}_M = \frac{q}{m}\rho_o(\vec{u}\times\vec{B})$ is the Lorentz force. This implies that the plasma is diluted enough so that the external field $\vec{B}$ will act only on the elementary charges $\pm q$ of the plasma. Here m is the mass of the point charge q.

Eqs. (1-2) can be linearized by introducing density and velocity fluctuations defined by:

$$\rho = \rho_0 + \delta\rho \tag{3}$$

$$\vec{u} = \vec{u}_0 + \delta\vec{u} \tag{4}$$

and,

$$\delta\theta \equiv \nabla \cdot (\delta\vec{u}) \tag{5}$$

where ρ_0 is the average density. The fluid is assumed to be static, so that $\vec{u}_0 = 0$, φ represents the gravitational potential and the external magnetic force is that corresponding to a constant magnetic field $\vec{B} = (B_0 + \delta B)\,\hat{k}$, so that the linearized equations for the density and velocity fluctuations can be written as:

$$\frac{\partial (\delta\rho)}{\partial t} + \rho_0 \delta\theta = 0 \tag{6}$$

and

$$\begin{aligned} \rho_0 \frac{\partial (\delta\vec{u})}{\partial t} &+ \nabla(\delta p) = -\rho_0 \nabla(\delta\varphi) \\ &+ \frac{q}{m}\rho_0 \left(\delta\vec{u} \times \vec{B}_0\right) \end{aligned} \tag{7}$$

where $\vec{f_g} = -\nabla(\delta\varphi)$, $\delta\varphi$ being the fluctuating gravitational potential. Neglecting temperature fluctuations, the pressure term in equation (7) may be rewritten in terms of the density fluctuations through the local equilibrium assumption namely, $p = p(\rho)$ so that

$$\nabla p = \left(\frac{\partial p}{\partial \rho_0}\right)_T \nabla\rho = \frac{c_0^2}{\gamma}\nabla\rho$$

We know recall that K_T, the thermal compressibility satisfies the relation $K_T = \gamma/c_0^2$ where $\gamma = C_p/C_v$ and c_0 is the velocity of sound in the medium. Eq. (7) may now be rewritten as

$$\begin{aligned} \rho_0 \frac{\partial\left(\delta\vec{u}\right)}{\partial t} &+ \frac{c_0^2}{\gamma}\nabla\left(\delta\rho\right) = -\rho_0\nabla\left(\delta\varphi\right) \\ &+ \frac{q}{m}\rho_0\left(\delta\vec{u}\times\vec{B_0}\right). \end{aligned} \tag{8}$$

Assuming that $\delta\varphi$ is given by Poisson's equation, $\nabla^2(\delta\varphi) = 4\pi G\delta\rho$, that $\vec{B}_o = B_o\hat{k}$ ($\hat{k}$ is the unit vector along the z-axis) and noticing that for this case the last term equals $B_o(\hat{j}u_x - \hat{i}u_y)$, equation (8) reduces to

$$\begin{aligned} -\rho_o\frac{\partial\left(\delta\theta\right)}{\partial t} &+ \frac{c_o^2}{\gamma}\nabla^2\left(\delta\rho\right) = -4\pi G\rho_o\left(\delta\rho\right) \\ &+ \frac{q}{m}\rho_o\left(\nabla\times\delta\vec{u}\right)_{\hat{k}} \end{aligned} \tag{9}$$

after taking its divergence and using equation (6).

Equations (6) and (9) are now two simultaneous equations for $\delta\rho$ and $\delta\vec{u}$ which need to be solved. To do so we introduce an approximation, namely that there exists cylindrical symmetry with respect to the z axis so that $\frac{\partial}{\partial z} = 0$. With this approximation the last term in equation (9) can be calculated taking the curl of equation (8). This yields

$$\begin{aligned} \rho_0\frac{\partial}{\partial t}\left(\nabla\times\delta\vec{u}\right)_{\hat{k}} &= \frac{q}{m}B_0\rho_0\nabla\times\left(\hat{j}\delta u_x - \hat{i}\delta u_y\right)_{\hat{k}} \\ &= -\frac{q}{m}B_0\rho_0\delta\theta. \end{aligned}$$

Finally, taking the time derivative of equation (9), using the previous equation and making use of Eq. (6), one is finally led to the result that

$$\begin{aligned} -\frac{\partial^3}{\partial t^3}\left(\delta\rho\right) &+ \frac{c_0^2}{\gamma}\nabla^2\left(\frac{\partial(\delta\rho)}{dt}\right) + 4\pi G\rho_0\left(\frac{\partial(\delta\rho)}{\delta t}\right) \\ &+ \left(\frac{qB_0}{m}\right)^2\frac{\partial}{\partial t}\left(\delta\rho\right) = 0 \end{aligned}$$

Integrating once with respect to time and setting the integrating constant equal to zero which deprives of no generality to our argument, we get that

$$\begin{aligned} -\frac{\partial^2}{\partial t^2}(\delta\rho) &+ \frac{c_0^2}{\gamma}\nabla^2(\delta\rho) + 4\pi G\rho_0(\delta\rho) \\ &+ \left(\frac{qB_0}{m}\right)^2(\delta\rho) = 0 \end{aligned} \tag{10}$$

Equation (10) is now a single equation for the density fluctuations $\delta\rho$. Its solution readily achieved proposing that $\delta\rho$ is described by a plane wave

$$\delta\rho = Ae^{i(\vec{k}\cdot\vec{r}-\omega t)}, \tag{11}$$

where all symbols are self defined. Substitution of equation (11) into equation (10) yields the dispersion relation,

$$\omega^2 - \frac{c_0^2}{\gamma}k^2 + 4\pi G\rho_0 + \left(\frac{qB_0}{m}\right)^2 = 0. \tag{12}$$

Instabilities enhanced by the gravitational and magnetic field arise when the roots for ω in this equation are imaginary. The threshold value of k beyond which this happens is precisely Jeans wave number and is here given by

$$k_J^2 = \frac{\gamma}{c_0^2}\left[4\pi G\rho_0 + \left(\frac{qB_0}{m}\right)^2\right]. \tag{13}$$

Clearly, if $B_0 = 0$ we recover the well known expression for k_J. The question now is how relevant is the second term in enhancing or hindering structure formation. This will be analyzed in the following section.

3 Analysis of the Dispersion Relation

As indicated in the previous section, the linearized version of fluctuating non-dissipative hydrodynamics, applied to a a dilute homogeneous plasma in the presence of a uniform magnetic field, leads to a Jeans wave number which is, as depicted in equation (13), a superposition of the gravitational and magnetic fields. The question is if this result has any bearing on the value of Jeans mass in realistic cases. Clearly equation (13) points at three possibilities, namely, if the magnetic term is a) negligible, b) of the same order or c) larger than the gravitational contribution. As we recall, Jeans mass is defined as

$$M_J \equiv \frac{4\pi}{3}\rho_o\lambda_J^3 = \frac{4\pi}{3}\rho_o\left(\frac{2\pi}{k_J}\right)^3$$

so that using equation (13)

$$M_J = \frac{32\pi^4}{3}\rho_o \left[\frac{C_o^2}{\gamma} \frac{1}{4\pi G\rho_o + \left(\frac{qB_o}{m}\right)^2} \right]^{3/2} \tag{14}$$

where $\rho_o = mn_o$, n_o being the particle density in the plasma. As it has been exhaustively discussed in the literature [4] [7] [8] [10] without the magnetic contribution, present values of $\rho_o \sim 10^{-29} gr/cm^3$, $m = m_H \sim 10^{-24} gr$ and $T \sim 10^5 K$ yield $M_J \sim 10^{11} M_\odot$.

Taking now q/m as the proton-mass ratio which is, in absolute value, approximately equal to $10^5 c/gr$, how does the magnetic term rates with respect to the gravitational one for different values of B_o^2?. Due to the uncertainties of the estimates for B_o according to current data [11] this may not be so easy to quantify.

According to the constraints imposed on cosmological magnetic fields from the different observations mentioned in the introduction, their strength seems to be confined in the range [11],

$$3 \times 10^{-8} \leq B \leq 10^{-6} G.$$

Taking $B = 10^{-6} G = 10^{-10} T$, the term $\frac{q}{m} B_o \sim 10^{-2} s^{-1}$ meaning that its squared value is $\sim 10^{-4} s^{-2}$. This would imply that $M_J \sim 1 M_\odot$, contrary to observations. Thus in agreement with Chandrasekhar and Fermi [3], the magnetic term is completely negligible insofar as the instabilities required for the formation of large structures, say galaxies. In order that $\frac{q}{m} B_o \sim 10^{-17} s^{-1}$, B_o must be of the order of $10^{-21} G$ which by the way is indeed of the order of magnitude of strengths that are envisaged to be detected at the cosmological level [11]. However, this is still a thought and not a fact so that expectation will remain until these measurements are achieved.

4 Conclusions

Even for the very simple case of a dilute not magnetized plasma in the presence of a homogeneous uniform magnetic field the value of Jeans's mass as predicted by gravitational instabilities seems not to be substantially modified according to our present knowledge of the values available for cosmological magnetic fields. Yet there is a possibility that much lower values could be observed in which case the value of M_J would be substantially modified. Thus, it appears that magnetic fields are probably not essential to our understanding of large-scale structures formation in the Universe.

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