Magnetic field effects in the heat flow of charged fluids: the Righi-Leduc effect.

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Heat conduction in ionized plasmas in the presence of magnetic fields is today a fashionable problem. The kinetic theory of plasmas, in the context of non-equilibrium thermodynamics, predicts a Hall-effect-like heat flow due to the presence of a magnetic field in ionized gases. This cross effect, the Righi-Leduc effect, is shown here to be, under certain conditions, more important than the Fourier component of the heat flow. The thermal conductivity associated with this effect grows with the strength of the magnetic field for a given temperature and density and is shown to be larger than the parallel conductivity for a whole range of values of $\vec{B}$.

The behavior of charged particles in the presence of external magnetic fields is a well known subject. According to the tenets of classical non-equilibrium thermodynamics for not too large gradients, the simultaneous heat and electrical flows in a magnetic field are linear functions of the temperature and electrical potential gradients, respectively. The coefficients appearing in these relations are, in general, second rank tensors which are functions of the external field and consist of a symmetrical part and an antisymmetrical part $^{1, 2, 3}$. The latter ones are called the “Hall vector” in the case of electrical conduction and the “Righi-Leduc” vector in the case of heat conduction. The former is a well known effect whereas the latter one isn’t. It was discovered in 1887 and first measured and confirmed by Waldemar Voigt in 1903 (see Ref. $^{2, 3}$). Curiously enough it has been, ever since, hardly mentioned in the literature. Not even the authors of a rather beautiful and striking experiment published recently $^{4}$ recognize that what they have really detected is the Righi-Leduc effect.

In this paper we address ourselves to a rather different aspect of this effect from the theoretical point of view. Indeed, in the case of a dilute ionized gas in the presence of weak magnetic fields for densities $n$ in the interval $10^{20} \leq n \leq 10^{22} \text{m}^{-3}$ and temperatures in the range $10^3 < T < 10^7 \text{K}$ the magnitude of heat, charge and mass currents arising from these and other so-called “cross effects” may contribute by factors several orders of magnitude larger than those arising from ordinary heat, mass and electrical conduction. This result seems to differ from those obtained by Balescu in his excellent and exhaustive treatment on plasma transport processes $^{5}$. We shall come back to this point a little later in this paper. Moreover, we believe that the approach here offered to the general subject of collisional transport processes in plasmas may provide a better understanding of this subject at temperatures and densities where collisions are favored in a non-relativistic framework.

The kinetic model we use in our approach to the problem is the well known one based on the Boltzmann equation as originally proposed by Chapman and Cowling $^{9}$. The substantial difference is that for the case of ionized gases they never developed their method up to the stage of placing it within the framework of linear irreversible thermodynamics. Although they derive
elementary expressions for the Righi–Leduc and the Nernst–Etinghausen effects (see Ref. [2]) they never pursued any comparison of their for-
mulae with experiments, much less study any astrophyiscal implications.

Thus, the system here considered is a binary mixture of electrically charged particles with masses and charges $m_i$ and $e_i$ for $i = a, b$. The density of the system is low enough such that the kinetic description is valid. For simplicity we shall set the charge of the ions $Z = 1$. For such a system, the evolution of the distribution functions of the molecules is given by the Boltzmann equation:

$$\frac{df_i}{dt} = \frac{\partial f_i}{\partial t} + \vec{v}_i \cdot \frac{\partial f_i}{\partial \vec{r}} + \frac{e_i}{m_i} \left( \vec{E} + \vec{v}_i \times \vec{B} \right) \cdot \frac{\partial f_i}{\partial \vec{v}_i} = \sum_{i,j=a}^b J(f_i f_j)$$

(1)

where the subscript $i$ indicates the species and $J(f_i f_j)$ is the collisional term representing collisions between different and same species. In the Lorentz force on the left side of Eq. (1) the electric and magnetic fields are consistent with Maxwell’s equations. The magnetic field is assumed to be small enough so that collisions dominate over cyclotron motion. This weak field approximation implies $\omega_i \tau \approx 1$ where $\omega_i = e_i B/m_i$ is the Larmor frequency, that is, the frequency of the circular orbits that the particles describe around magnetic field lines.

From Eq. (1) the derivation of the conservation equations as well as the proof of the $H$-theorem are logically required. For reasons of space we shall content ourselves asserting that both steps have been completely fulfilled but we leave the details for a much longer forthcoming publication [3]. We here proceed directly with the solution of Eq. (1) following the standard Hilbert–Chapman–Enskog approximation. Since a local Maxwellian distribution function is clearly a solution to the homogeneous part of Eq. (1) we propose that the single particle distribution functions $f_i$ ($i = a, b$) may be expanded in a power series of Knudsen’s parameter $\epsilon$ which, as well known, is a measure of the magnitude of the macroscopic gradients [6, 7, 9].

Thus

$$f_i = f_i^{(0)} + \epsilon \varphi_i^{(1)} + \mathcal{O}(\epsilon^2)$$

(2)

In Eq. (2) the local equilibrium assumption is also invoked, namely the time dependence of $f_i^{(0)} \varphi_i^{(n)}$ for all $n$ occurs only through the conserved densities. The particle density is $n_i(\vec{r}, t)$ ($i = a, b$), the barycentric velocity $\vec{u}(\vec{r}, t)$ and the local temperature $T(\vec{r}, t)$ is in this paper assumed to be the same for ions and electrons [10]. $T(\vec{r}, t)$ is related to the internal energy density $\varepsilon(\vec{r}, t)$ by the standard ideal gas relationship. In this work we shall deal only with the first order in the gradients correction to $f_i$, namely the Navier–Stokes–Fourier regime.

Substitution of Eq. (2) into Eq. (1), leads to order zero in $\epsilon$, to the Euler equations of magnetohydrodynamics. To first order in $\epsilon$ one obtains a set of two linear integral equations for $\varphi_i^{(1)}$ which involve the linearized collision kernels in their homogeneous terms whereas the inhomogeneous ones contain a combination of terms involving the macroscopic gradients $\nabla T$, $\nabla \vec{u}$ and the diffusive force $\vec{d}_{ij} = -\vec{d}_{ji}$. We must emphasize that, as has been recently shown [11], it is only in this representation that the Onsager reciprocity relations hold true, at least when $\vec{B} = 0$. If $\vec{B}$ is different from zero the proof of such relations starting from the linearized integral equations for $\varphi_i^{(1)}$ will be discussed elsewhere. The solution to these equations arises from the standard procedure [6, 9, 2] and is shown to be of the form

$$\varphi_i^{(1)} = -\vec{\alpha}_j \cdot \frac{\nabla T}{T} - \vec{\beta}_i \cdot \vec{d}_{ij}$$

(3)

where $\vec{\alpha}_j$ and $\vec{\beta}_i$ are the most general vectors that can be constructed from the vectors $\vec{c}_i$, $\vec{c}_i \times \vec{B}$ and $\left( \vec{c}_i \times \vec{B} \right) \times \vec{B}$. A term $\vec{B} : \nabla u$ has been neglected here for simplicity but will be taken into account in a separate work where the viscomagnetic effects are to be dealt with.
Here
\[\delta_{ab}^\text{e} = \frac{n_a}{n} + \frac{n_a n_b (m_a - m_b)}{n \rho} \nabla p - \frac{n_a n_b}{\rho p} (m_b \epsilon_a - m_a \epsilon_b) \cdot \mathbf{E}' - d_\text{e}^\text{a} \]  

where \(\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}\) is the effective electric field seen by an observer moving with the barycentric velocity \(\mathbf{u}\). Also \(n = n_a + n_b\), \(\rho = m_a n_a + m_b n_b\) and \(p\) is the local hydrodynamic pressure. Clearly then
\[\tilde{\mathbf{k}}_a = \mathbf{A}_a^{(1)} \tilde{\epsilon}_a + \mathbf{A}_a^{(2)} (\tilde{\epsilon}_a \times \mathbf{B}) + \mathbf{A}_a^{(3)} \tilde{\mathbf{B}} \times (\tilde{\epsilon}_a \times \tilde{\mathbf{B}})\]  

and a similar expression for \(\tilde{\mathbf{k}}_b\). The term \(\tilde{\epsilon}_a \times \mathbf{B}\) arising from the Lorentz force is now present in the right hand side of the linearized version of Eq. (1) and thus contains a direct influence of the magnetic field on the inter-particle collisions \((\mathbf{u} \ll \tilde{\epsilon}_a)\) (see Ref. [3]).

Let us now restrict ourselves to examine the nature of the heat conduction term in Eq. (2). A similar discussion can be carried over to the diffusive contributions to the heat flow but we shall leave them out for the sake of clarity. The heat flux in this system is simply given by
\[\tilde{J}_Q = k^2 T \sum_{i=a}^b \int \left( \frac{m c^2}{2kT} - \frac{5}{2} \right) f_i^{(0)} \varphi_i^{(1)} \tilde{\epsilon}_i d\tilde{\epsilon}_i\]  

Substituting Eqs. (3) and (4) into Eq. (5) and using the Sonine polynomial representation for the unknown functions \(\mathbf{A}_k^{(j)} (j = 1, 2, 3)\) namely
\[\mathbf{A}_k^{(j)} = \sum_{m=0}^{\infty} a_k^{(j)(m)} \varphi_{3/2}^{(m)} (\epsilon_i^2)\]  

one obtains after somewhat lengthy and cumbersome algebra [7] that Eq. (6) reads as
\[\tilde{J}_Q = -\frac{5}{2} k^2 T \left\{ \kappa \| \nabla \| T + \kappa_\perp \nabla_\perp T + \kappa_R \nabla_s T \right\}\]  

This is the first important result of this work. To appreciate it let’s fix the direction of the magnetic field along the z-axis, \(\tilde{\mathbf{B}} = \mathbf{B}k\) and take cartesian coordinates. Then \((\nabla T)_\perp = \frac{\partial T}{\partial x} \hat{i} + \frac{\partial T}{\partial y} \hat{j}\) is a vector in the x-y plane, perpendicular to \(\mathbf{B}\). The vector \((\nabla T)_\parallel = \frac{\partial T}{\partial z} \hat{k}\) is parallel to \(\tilde{\mathbf{B}}\). The term \((\nabla T)_R = k \times \nabla T = -\frac{\partial T}{\partial x} \hat{i} + \frac{\partial T}{\partial y} \hat{j}\) is a vector perpendicular to both \(\tilde{\mathbf{B}}\) and \(\nabla T\). The corresponding heat flux is precisely the Right-Leduc effect. Notice that if \(\tilde{\mathbf{B}}\) is constant \(\nabla \cdot (\tilde{\mathbf{B}} \times \nabla T) = 0\) so the heat current occurs in spirals [see Ref. [2]]. But if \(\tilde{\mathbf{B}}\) is inhomogeneous, which may be the case since all the operations leading to Eq. (8) have been performed in the velocity space, then a new term arises \(\nabla T \cdot (\nabla \times \tilde{\mathbf{B}})\), the ensuing temperature profile is coupled to Maxwell’s equations through \(\nabla \times \tilde{\mathbf{B}}\). We therefore have two rather important results
\[\tilde{J}_{QF} = -\kappa_\parallel \nabla T\]  

the ordinary, Fourier, heat conduction and two additional contributions, a heat flow in the direction perpendicular to the magnetic field with a conductivity \(\kappa_\perp\) and the heat flow corresponding to the Right-Leduc effect. If \(\tilde{\mathbf{B}} = 0\) the latter vanishes and \(\kappa_\perp = \kappa_\parallel\). To assess the importance of these effects, we need to solve a set of linear integral equations for the functions \(\mathbf{A}_k^{(j)}\) to calculate the appropriate coefficients \(a_k^{(j)(m)} (k = a, b\) and \(j = 1, 2, 3)\). This involves a lengthy, tedious calculation which may be performed [7] using a modification of Hirschfelder’s variational method [12] as implemented by Ziman [13]. The results are: for the thermal conductivity one obtains
\[\kappa_\parallel = \frac{30 \pi^{3/2} k (kT)^{5/2} \epsilon_0^2}{\sqrt{m_e e^4 \psi}} \times 0.0875 \frac{J}{s K m}\]  

where
\[\psi = \ln \left[ 1 + \left( \frac{16 \pi k \lambda \epsilon_0}{e^2} \right)^2 \right]\]  

and
\[ \lambda_d = \sqrt{\frac{\epsilon_0 kT}{n e^2}} \]  

(12)

is the well known Debye’s length. Also, \( k \) is Boltzmann’s constant and \( \epsilon_0 = 8.854 \times 10^{-12} \, F/m^{-1} \). The conductivities \( \kappa_\perp \) and \( \kappa_R \) have the same form as Eq. (10) except that the numerical factor is now substituted by functions \( g(x) \) and \( h(x) \), respectively which are the ratio of two polynomials in \( x = \omega_c \tau \). Indeed, \( h(x) \) in explicitly given by

\[ g(x) = \frac{-1.4 \times 10^3 + 5.1 \times 10^4 x^2 - 5.4 \times 10^2 x^4}{1.6 \times 10^4 + 3.2 \times 10^5 x^2 + 10^6 x^4 + 4.7 x^6} \]  

and

\[ h(x) = \frac{3.5 \times 10^4 x + 2.5 \times 10^5 x^3 + 2.3 x^5}{1.6 \times 10^4 + 3.2 \times 10^5 x^2 + 10^6 x^4 + 4.7 x^6} \]  

(13)

(14)

In Eqs. (13-14) \( \tau \) is the mean free time extracted directly from the collision integrals involved in the previous calculation

\[ \tau = \frac{24 \pi^{3/2} \sqrt{m_e} (kT)^{3/4} \epsilon_0^2}{ne^4 \psi} \]  

(15)

and as said before, \( \omega_c = eB/m_e \), \( m_e \) being the mass of the electron. Equation (10) differs from the result obtained from the Spitzer-Bragnis formula [1] at most by a factor of 3.6%. The surprising results are shown in Fig 1 for thermodynamical parameters typical of accretion disks in binary systems.

Figure 1 shows the difference between the parallel and the Righi-Leduc thermal conductivities as a function of the magnetic field for a temperature of \( T = 10^7 K \) and density of \( n = 10^{21} m^{-3} \), as found in accretion disks in binary systems [12]. The interval in \( B \) is chosen such that the \( \omega \tau \approx 1 \) condition is satisfied. It can be seen that \( \kappa_R \) increases from zero until a critical value of \( B \) where \( \kappa_R \) becomes greater than the parallel thermal conductivity.

It is important to emphasize the fact that for these densities and \( T = 10^7 K \) the magnetic field for which the approximation \( \omega_c \tau \approx 1 \) holds true ranges between \( 10^2 \) and \( 10^4 \mu G \). We recall that \( \tau \), given in Eq. (15), is a function of \( n \) and \( T \) so that this approximation has to be handled with care. At this stage it is convenient to compare our results with those obtained by Balescu in Ref. [3]. Although he uses the Landau version of the collision term instead of the full Boltzmann equation, there is a difference between his result and ours as seen when Fig. 1 is compared with Fig. 5.1 in Balescu’s textbook [3]. Qualitatively the behavior of the three thermal conductivities is the same, the striking difference is that the Righi-Leduc conduc-
tivity, in his case never exceeds the parallel one, whereas in our case under the conditions specified, becomes approximately seven times larger. We believe that the reason is due to the fact that he uses the Grad-like moment expansion to solve the kinetic equation and, as it has been clearly pointed out by one of us [13, 16], this implies some confusion as to which are the contributions to a given specific order in the gradients to a transport coefficient arising from that approach. A more detailed comparison will be given elsewhere. Finally we wish to remark also that this result together with a similar one obtained with the Dufour coefficient may be useful in accounting for dissipative phenomena which are becoming rather important in the physics of the intracluster medium [18, 19]. This work has been supported by CONACyT Mexico.

[10] No difficulties would arise in assuming that the ion temperature $T_i$ is not equal to the electron temperature $T_e$, it would only complicate the algebra. For a discussion see S. I. Braginski; Rev. of Plasma Physics Vol. I; Consultants Bureau N. Y. 1965 pp. 205.